# Regular Expressions

# Recap from Last Time

# Regular Languages

- A language $L$ is a ***regular language*** iff there is a DFA $D$ such that $\mathscr{L}(D) = L$.

- ***Theorem:*** The following are equivalent:
  - $L$ is a regular language.
  - There is a **DFA** for $L$.
  - There is an **NFA** for $L$.

# Language Concatenation

- If $w \in \Sigma^*$ and $x \in \Sigma^*$, then $wx$ is the ***concatenation*** of $w$ and $x$.
- If $L_1$ and $L_2$ are languages over $\Sigma$, the ***concatenation*** of $L_1$ and $L_2$ is the language $L_1L_2$ defined as

$$\boldsymbol{L_1L_2 = \{ wx \mid w \in L_1 \text{ and } x \in L_2 \}}$$

- Example: if $L_1 = \{$ **a**, **ba**, **bb** $\}$ and $L_2 = \{$ **aa**, **bb** $\}$, then

$L_1L_2 = \{$ **aaa**, **abb**, **baaa**, **babb**, **bbaa**, **bbbb** $\}$

# Lots and Lots of Concatenation

- Consider the language $L$ = { **aa**, **b** }
- $L^0 = \{\varepsilon\}$
- $LL=L^2$ is the set of strings formed by concatenating pairs of strings in $L$.

{ **aaaa**, **aab**, **baa**, **bb** }

- $LLL = L^3$ is the set of strings formed by concatenating triples of strings in $L$.

{ **aaaaaa**, **aaaab**, **aabaa**, **aabb**, **baaaa**, **baab**, **bbaa**, **bbb**}

- $LLLL = L^4$ is the set of strings formed by concatenating quadruples of strings in $L$.

{ **aaaaaaaa**, **aaaaaab**, **aaaabaa**, **aaaabb**, **aabaaaa**, **aabaab**, **aabbaa**, **aabbb**, **baaaaaa**, **baaaab**, **baabaa**, **baabb**, **bbaaaa**, **bbaab**, **bbbaa**, **bbbb**}

# The Kleene Closure

- An important operation on languages is the ***Kleene Closure***, which is defined as

$$L^* = \{\ w \in \Sigma^* \mid \exists n \in \mathbb{N}.\ w \in L^n\ \}$$

# Closure Properties

- ***Theorem:*** If $L_1$ and $L_2$ are regular languages over an alphabet $\Sigma$, then so are the following languages:
  - $\overline{L_1}$
  - $L_1 \cup L_2$
  - $L_1 \cap L_2$
  - $L_1L_2$
  - $L_1*$
- These properties are called ***closure properties of the regular languages***.

**Quick check 1:**
Let $\Sigma$ = {1, 2, 3, a, b, c}.
Let $L_1$ = {aa, b}, $L_2$ = {33, 2}
be languages over $\Sigma$.

Name one string **in** $\overline{L_1}$.
Name one string **not in** $\overline{L_1}$.

# Closure Properties

- ***Theorem:*** If $L_1$ and $L_2$ are regular languages over an alphabet Σ, then so are the following languages:
  - $\overline{L_1}$
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  - $L_1^*$
- These properties are called ***closure properties of the regular languages***.

**Quick check 2:**
Let Σ = {1, 2, 3, a, b, c}.
Let $L_1$ = {aa, b}, $L_2$ = {33, 2}
be languages over Σ.

Name one string **in** $L_1 \cup L_2$.
Name one string **not in** $L_1 \cup L_2$.

# Closure Properties

- ***Theorem:*** If $L_1$ and $L_2$ are regular languages over an alphabet Σ, then so are the following languages:
  - $\overline{L_1}$
  - $L_1 \cup L_2$
  - $L_1 \cap L_2$
  - $L_1L_2$
  - $L_1*$
- These properties are called ***closure properties of the regular languages***.

**Quick check 3:**
Let Σ = {1, 2, 3, a, b, c}.
Let $L_1$ = {aa, b}, $L_2$ = {33, 2}
be languages over Σ.

Name one string **in** $L_1 \cap L_2$.
Name one string **not in** $L_1 \cap L_2$.

# Closure Properties

- ***Theorem:*** If $L_1$ and $L_2$ are regular languages over an alphabet Σ, then so are the following languages:
  - $\overline{L_1}$
  - $L_1 \cup L_2$
  - $L_1 \cap L_2$
  - $L_1L_2$
  - $L_1^*$
- These properties are called ***closure properties of the regular languages***.

**Quick check 4:**
Let Σ = {1, 2, 3, a, b, c}.
Let $L_1$ = {aa, b}, $L_2$ = {33, 2}
be languages over Σ.

Name one string **in** $L_1L_2$.

Name one string **not in** $L_1L_2$.

# Closure Properties

- ***Theorem:*** If $L_1$ and $L_2$ are regular languages over an alphabet Σ, then so are the following languages:
  - $\overline{L_1}$
  - $L_1 \cup L_2$
  - $L_1 \cap L_2$
  - $L_1L_2$
  - $L_1^*$
- These properties are called ***closure properties of the regular languages***.

**Quick check 5:**
Let Σ = {1, 2, 3, a, b, c}.
Let $L_1$ = {aa, b}, $L_2$ = {33, 2}
be languages over Σ.

Name one string **in** $L_1^*$.

Name one string **not in** $L_1^*$.

# New Stuff!

# Another View of Regular Languages

# Rethinking Regular Languages

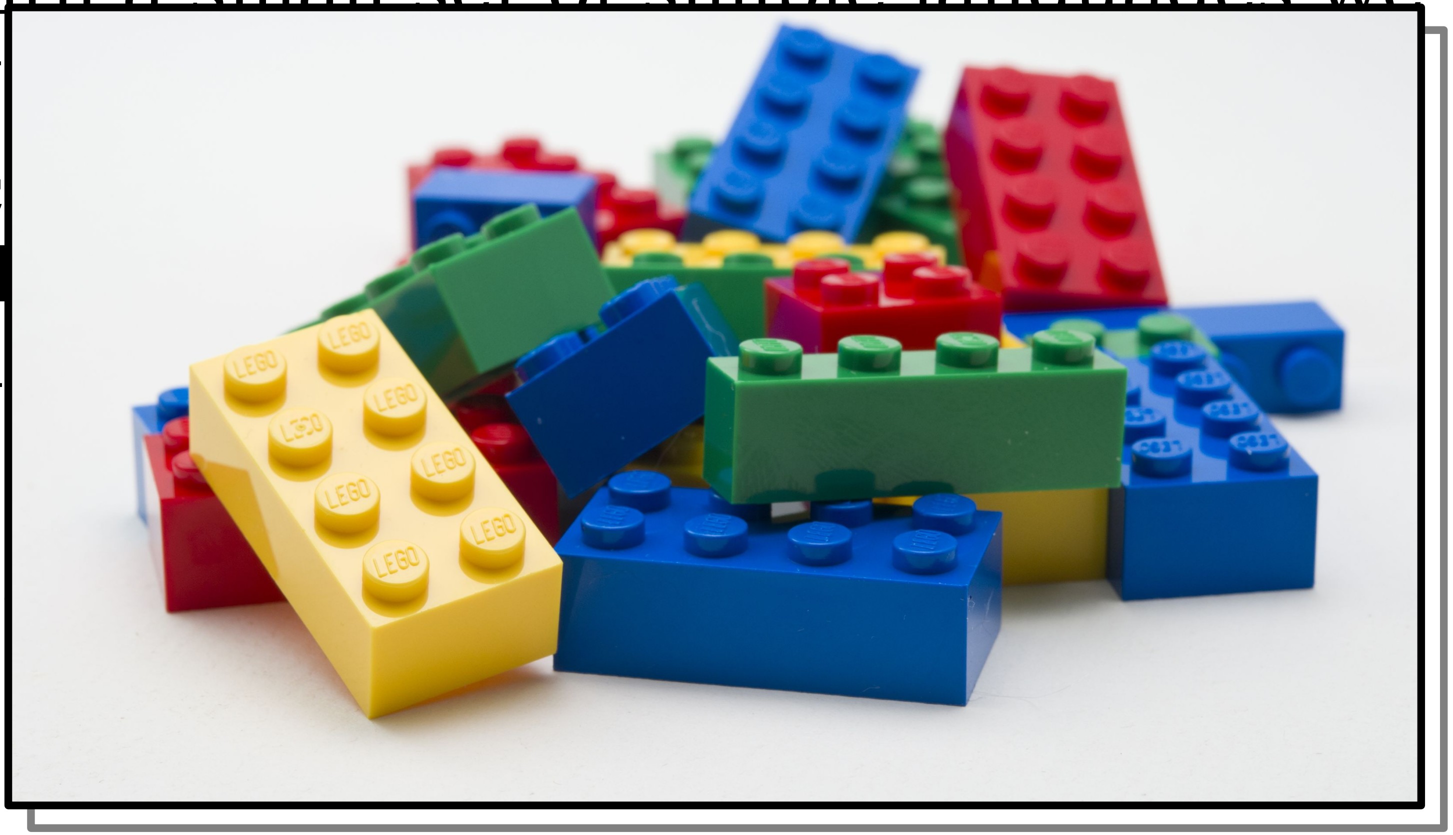
- We currently have several tools for showing a language $L$ is regular:
  - Construct a **DFA** for $L$.
  - Construct an **NFA** for $L$.
  - Combine several simpler regular languages together via **closure properties** to form $L$.
- Today we expand on this last idea.

# Constructing Regular Languages

- ***Idea:*** Build up all regular languages as follows:
  - Start with a small set of simple languages we already know to be regular.
  - Using closure properties, combine these simple languages together to form more elaborate languages.
- *This is a bottom-up approach to the regular languages.*

# Constructing Regular Languages

- ***Idea:*** Build up all regular languages as follows:
  - Start with a small set of simple languages we already
  - Using c
    simple l
    elabora
- *This is a*
  *regular l*



# Regular Expressions

- ***Regular expressions*** are a way of describing a language via a string representation.
- They're used just about everywhere:
  - They're built into the JavaScript language and used for data validation.
  - They're used in the UNIX `grep` and `flex` tools to search files and build compilers.
  - They're employed to clean and scrape data for large-scale analysis projects.
- Conceptually, regular expressions are strings describing how to assemble a larger language out of smaller pieces.

# Atomic Regular Expressions

- The regular expressions begin with three simple building blocks.
- The symbol **Ø** is a regular expression that represents the empty language Ø.
- For any **a** ∈ Σ, the symbol **a** is a regular expression for the language {**a**}.
- The symbol **ε** is a regular expression that represents the language {ε}.
  - ***Remember: {ε} ≠ Ø!***
  - ***Remember: {ε} ≠ ε!***

# Compound Regular Expressions

- If $R_1$ and $R_2$ are regular expressions, $\boldsymbol{R_1R_2}$ is a regular expression for the ***concatenation*** of the languages of $R_1$ and $R_2$.
- If $R_1$ and $R_2$ are regular expressions, $\boldsymbol{R_1 \cup R_2}$ is a regular expression for the ***union*** of the languages of $R_1$ and $R_2$.
- If $R$ is a regular expression, $\boldsymbol{R^*}$ is a regular expression for the ***Kleene closure*** of the language of $R$.
- If $R$ is a regular expression, $\boldsymbol{(R)}$ is a regular expression with the same meaning as $R$.

# Operator Precedence

- Here's the operator precedence for regular expressions:

$$(R)$$

$$R^*$$

$$R_1R_2$$

$$R_1 \cup R_2$$

- So **ab*c∪d** is parsed as **((a(b*))c)∪d**

# Regular Expression Examples

- The regular expression **trick∪treat** represents the language

  { **trick**, **treat** }.

- The regular expression **booo*** represents the regular language

  { **boo**, **booo**, **boooo**, … }.

- The regular expression **candy!(candy!)*** represents the regular language

  { **candy!**, **candy!candy!**, **candy!candy!candy!**, … }.

# Regular Expressions, Formally

- The ***language of a regular expression*** is the language described by that regular expression.
- Formally:
  - $\mathscr{L}(\boldsymbol{\varepsilon}) = \{\varepsilon\}$
  - $\mathscr{L}(\boldsymbol{\emptyset}) = \emptyset$
  - $\mathscr{L}(\mathbf{a}) = \{\mathbf{a}\}$
  - $\mathscr{L}(R_1R_2) = \mathscr{L}(R_1)\ \mathscr{L}(R_2)$
  - $\mathscr{L}(R_1 \cup R_2) = \mathscr{L}(R_1) \cup \mathscr{L}(R_2)$
  - $\mathscr{L}(R\mathbf{*}) = \mathscr{L}(R)^*$
  - $\mathscr{L}(\mathbf{(}R\mathbf{)}) = \mathscr{L}(R)$

Worthwhile activity: Apply this recursive definition to

**a(b∪c)((d))**

and see what you get.

# Regular Expressions, Formally

- The ***language of a regular expression*** is the language described by that regular expression.
- Formally:
  - $\mathscr{L}(\boldsymbol{\varepsilon}) = \{\varepsilon\}$
  - $\mathscr{L}(\boldsymbol{\emptyset}) = \emptyset$
  - $\mathscr{L}(\mathbf{a}) = \{\mathbf{a}\}$
  - $\mathscr{L}(R_1R_2) = \mathscr{L}(R_1)\,\mathscr{L}(R_2)$
  - $\mathscr{L}(R_1 \cup R_2) = \mathscr{L}(R_1) \cup \mathscr{L}(R_2)$
  - $\mathscr{L}(R^*) = \mathscr{L}(R)^*$
  - $\mathscr{L}((R)) = \mathscr{L}(R)$

**Regex quick check:**
Let $\Sigma = \{a, b, c, d\}$.
Let $L_1 = \mathcal{L}(\mathbf{a(b \cup c)((d))})$ be a language over $\Sigma$.

Name one string **in** $L_1$.
Name one string **not in** $L_1$.

# Designing Regular Expressions

- Let $\Sigma = \{\mathbf{a}, \mathbf{b}\}$.
- Let $L = \{ w \in \Sigma^* \mid w$ contains **aa** as a substring $\}$.

# Designing Regular Expressions

- Let $\Sigma$ = {**a**, **b**}.
- Let $L$ = { $w \in \Sigma^*$ | $w$ contains **aa** as a substring }.

**(a ∪ b)*aa(a ∪ b)***

# Designing Regular Expressions

- Let $\Sigma = \{\mathbf{a}, \mathbf{b}\}$.
- Let $L = \{ w \in \Sigma^* \mid w \text{ contains } \mathbf{aa} \text{ as a substring} \}$.

$$(\mathbf{a} \cup \mathbf{b})^*\mathbf{aa}(\mathbf{a} \cup \mathbf{b})^*$$

# Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{\ w \in \Sigma^* \mid w$ contains **aa** as a substring $\}$.

**(a ∪ b)*aa(a ∪ b)***

**bbabbbaabab**

**aaaa**

**bbbbabbbbaabbbbb**

# Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid w$ contains **aa** as a substring $\}$.

**(a ∪ b)*aa(a ∪ b)***

**bbabbbaabab**

**aaaa**

**bbbbbabbbbaabbbb**

# Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid w$ contains **aa** as a substring $\}$.

**Σ*aaΣ***

**bbabbbaabab**

**aaaa**

**bbbbabbbbaabbbb**

# Designing Regular Expressions

- Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$.
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

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Let $\Sigma = \{\mathbf{a}, \mathbf{b}\}$.

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The length of a string *w* is denoted $|w|$

# Designing Regular Expressions

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- Let $\Sigma = \{$**a**, **b**$\}$.
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**ΣΣΣΣ**

# Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

$\Sigma\Sigma\Sigma\Sigma$

# Designing Regular Expressions

- Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$.
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

**ΣΣΣΣ**

**aaaa**
**baba**
**bbbb**
**baaa**

# Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

**ΣΣΣΣ**

**aaaa**
**baba**
**bbbb**
**baaa**

# Designing Regular Expressions

- Let $\Sigma = \{\mathbf{a}, \mathbf{b}\}$.
- Let $L = \{ w \in \Sigma^* \mid |w| = 4 \}$.

$\Sigma^4$

aaaa

baba

bbbb

baaa

# Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{\ w \in \Sigma^* \mid |w| = 4\ \}$.

$\Sigma^4$

**aaaa**
**baba**
**bbbb**
**baaa**

# Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{\ w \in \Sigma^* \mid w$ contains at most one **a** $\}$.

Here are some candidate regular expressions for the language $L$. **How many** of these are correct? (Discuss specifically which with your neighbors.)

**Σ*aΣ***

**b*ab* ∪ b***

**b*(a ∪ ε)b***

**b*a*b* ∪ b***

**b*(a* ∪ ε)b***

# Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid w$ contains at most one **a** $\}$.

**b*(a ∪ ε)b***

# Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid w$ contains at most one **a** $\}$.

**b*(a ∪ ε)b***

# Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{\ w \in \Sigma^* \mid w$ contains at most one **a** $\}$.

**b★(a ∪ ε)b★**

**bbbbabbb**

**bbbbbb**

**abbb**

**a**

# Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{ w \in \Sigma^* \mid w$ contains at most one **a** $\}$.

**b*(a ∪ ε)b***

**bbbbabbb**

**bbbbbb**

**abbb**

**a**

# Designing Regular Expressions

- Let $\Sigma = \{$**a**, **b**$\}$.
- Let $L = \{\ w \in \Sigma^* \mid w$ contains at most one **a** $\}$.

**b*a?b***

**bbbbabbb**

**bbbbbb**

**abbb**

**a**

# A More Elaborate Design

- Let $\Sigma$ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

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**cs103@cs.stanford.edu**

**first.middle.last@mail.site.org**

**dot.at@dot.com**

# A More Elaborate Design

- Let $\Sigma$ = { **a**, **.**, **@** }, where **a** represents "some letter."
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**cs103**@cs.stanford.edu

**first**.middle.last@mail.site.org

**dot**.at@dot.com

# A More Elaborate Design

- Let $\Sigma$ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

```
aa*
```

```
cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com
```

# A More Elaborate Design

- Let $\Sigma$ = { **a**, **.**, **@** }, where **a** represents "some letter."
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```
aa*
```

```
cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com
```

# A More Elaborate Design

- Let $\Sigma$ = { `a`, `.`, `@` }, where `a` represents "some letter."
- Let's make a regex for email addresses.

```
aa*(.aa*)*
```

```
cs103@cs.stanford.edu
first.middle.last@mail.site.org
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```

# A More Elaborate Design

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```
aa*(.aa*)*
```

```
cs103@cs.stanford.edu
first.middle.last@mail.site.org
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```

# A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

```
aa*(.aa*)*@
```

```
cs103@cs.stanford.edu
first.middle.last@mail.site.org
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```

# A More Elaborate Design

- Let $\Sigma$ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

```
aa*(.aa*)*@
```

```
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```

# A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

```
aa*(.aa*)*@aa*.aa*
```

```
cs103@cs.stanford.edu
first.middle.last@mail.site.org
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```

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- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

```
aa*(.aa*)*@aa*.aa*
```

```
cs103@cs.stanford.edu
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```

# A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

**aa*(.aa*)*@aa*.aa*(.aa*)***

**cs103@cs.stanford.edu**

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# A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

```
aa*(.aa*)*@aa*.aa*(.aa*)*
```

```
cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com
```

# A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

$$\mathtt{a}^{+}\mathtt{(.aa^{*})^{*}@aa^{*}.aa^{*}(.aa^{*})^{*}}$$

**cs103@cs.stanford.edu**

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# A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
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$$a^+(.aa^*)^*@aa^*.aa^*(.aa^*)^*$$

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# A More Elaborate Design

- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

$$\texttt{a}^+ \ (\texttt{.a}^+)^* \ \texttt{@} \ \ \texttt{a}^+ \ \texttt{.a}^+ \ (\texttt{.a}^+)^*$$

```
cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com
```

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- Let Σ = { **a**, **.**, **@** }, where **a** represents "some letter."
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```
a+ (.a+)* @ a+.a+ (.a+)*
```

```
cs103@cs.stanford.edu
first.middle.last@mail.site.org
dot.at@dot.com
```

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- Let $\Sigma$ = { **a**, **.**, **@** }, where **a** represents "some letter."
- Let's make a regex for email addresses.

$$\mathtt{a^+\ (.a^+)^*\ @\ a^+\ .a^+\ (.a^+)^*}$$

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- Let's make a regex for email addresses.

$$\mathtt{a^+ \ (.a^+)^\star \ @ \ a^+ \ (.a^+)^+}$$

**cs103@cs.stanford.edu**

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$$\mathtt{a^+(.a^+)^*@a^+(.a^+)^+}$$

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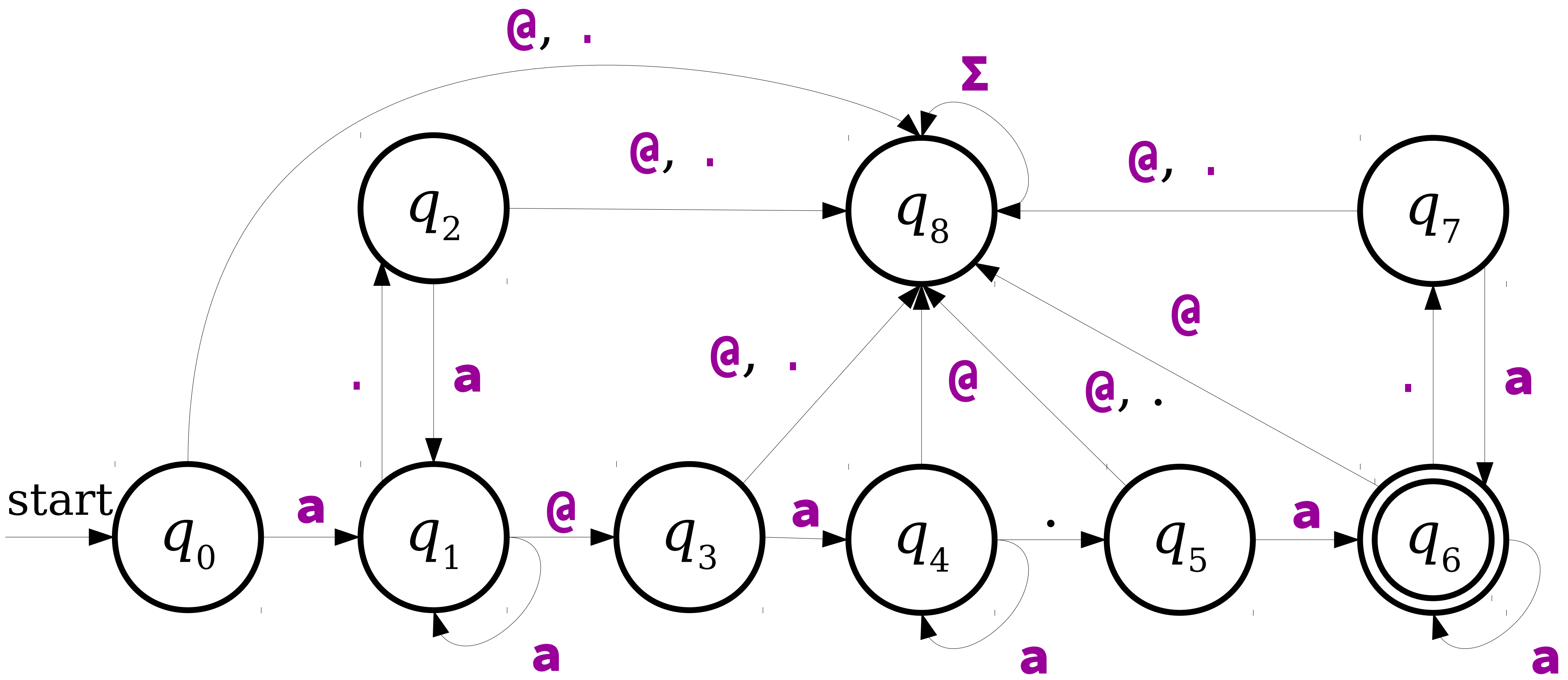
**dot.at@dot.com**

# For Comparison

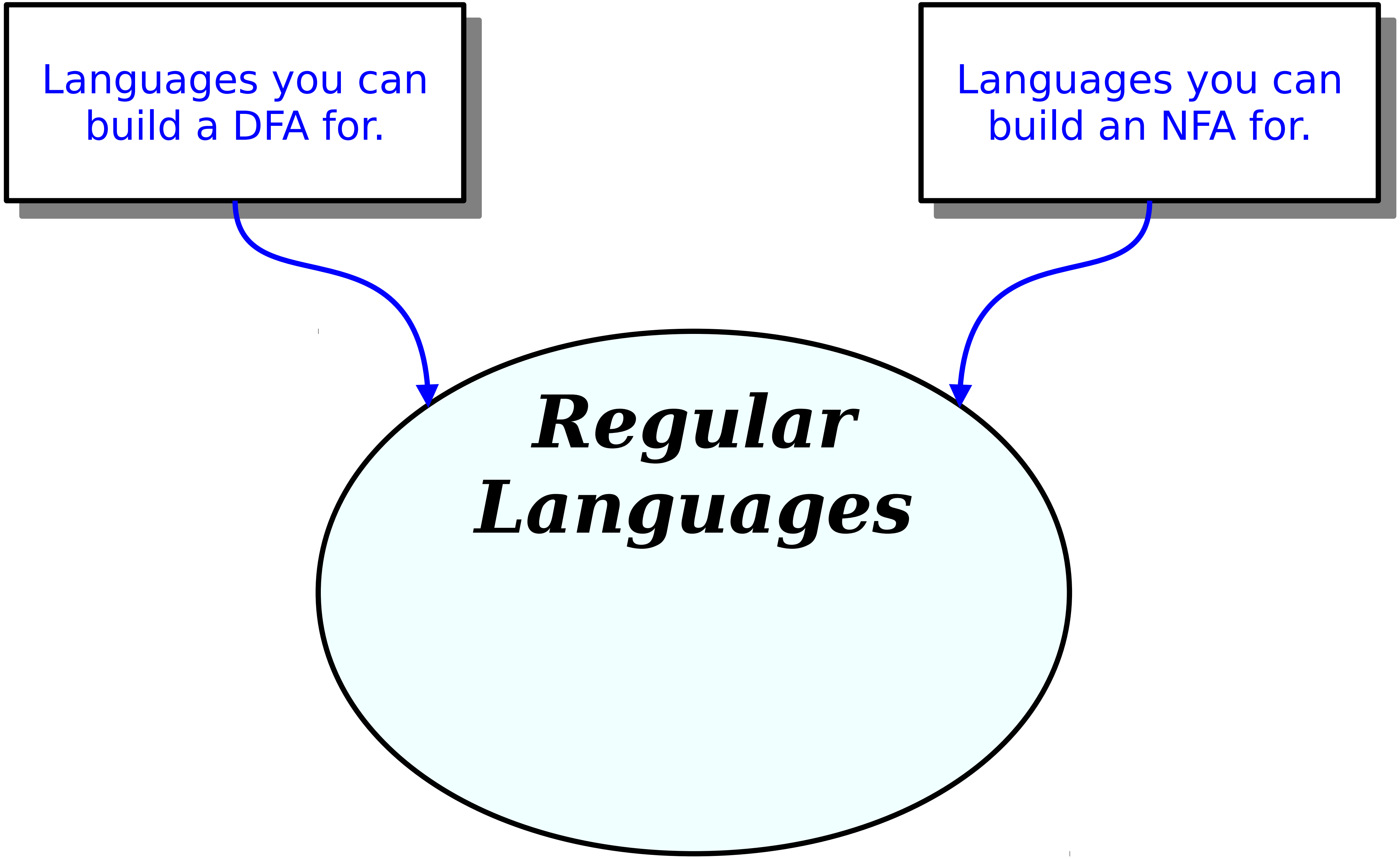
**a⁺(.a⁺)*@a⁺(.a⁺)⁺**

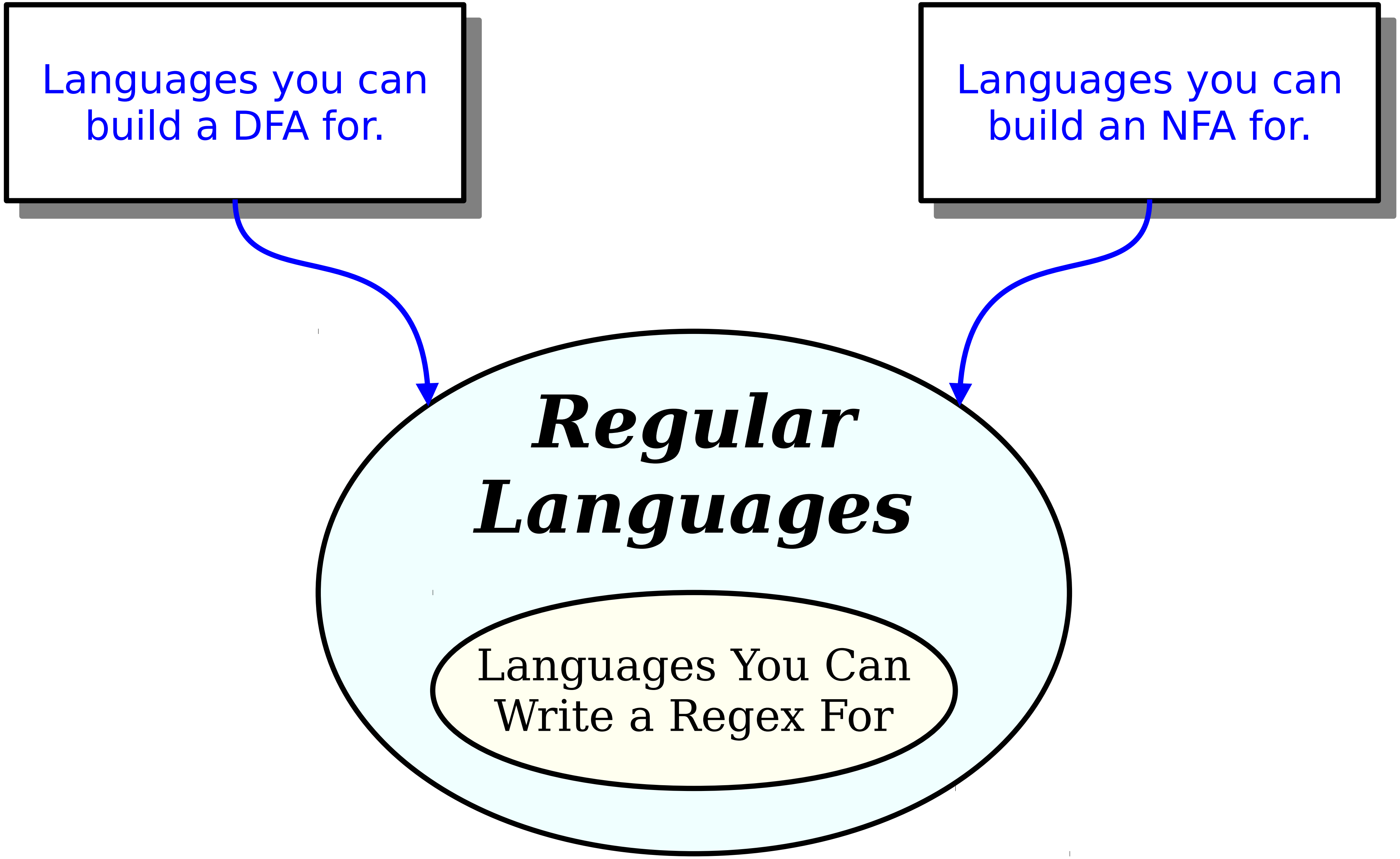


# Shorthand Summary

- $\mathbf{R}^n$ is shorthand for **RR … R** ($n$ times).
  - Edge case: define $\mathbf{R}^0 = \varepsilon$.
- **Σ** is shorthand for "any character in Σ."
- **R?** is shorthand for **(R ∪ ε)**, meaning "zero or one copies of *R*."
- $\mathbf{R}^+$ is shorthand for **RR★**, meaning "one or more copies of *R*."

# The Lay of the Land


Languages you can build a DFA for.
Languages you can build an NFA for.
Regular Languages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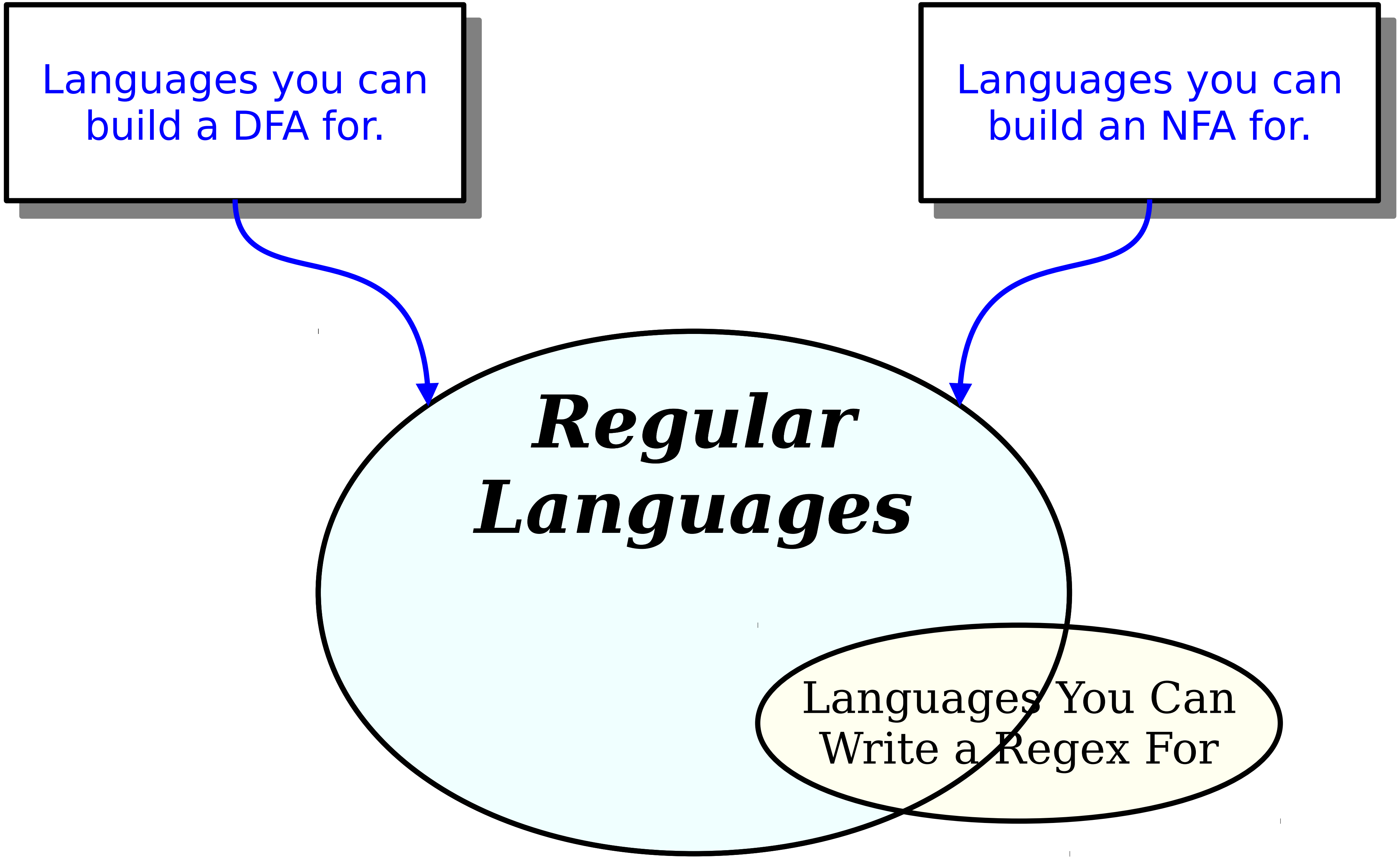
Regular Languages
Languages You Can Write a Regex For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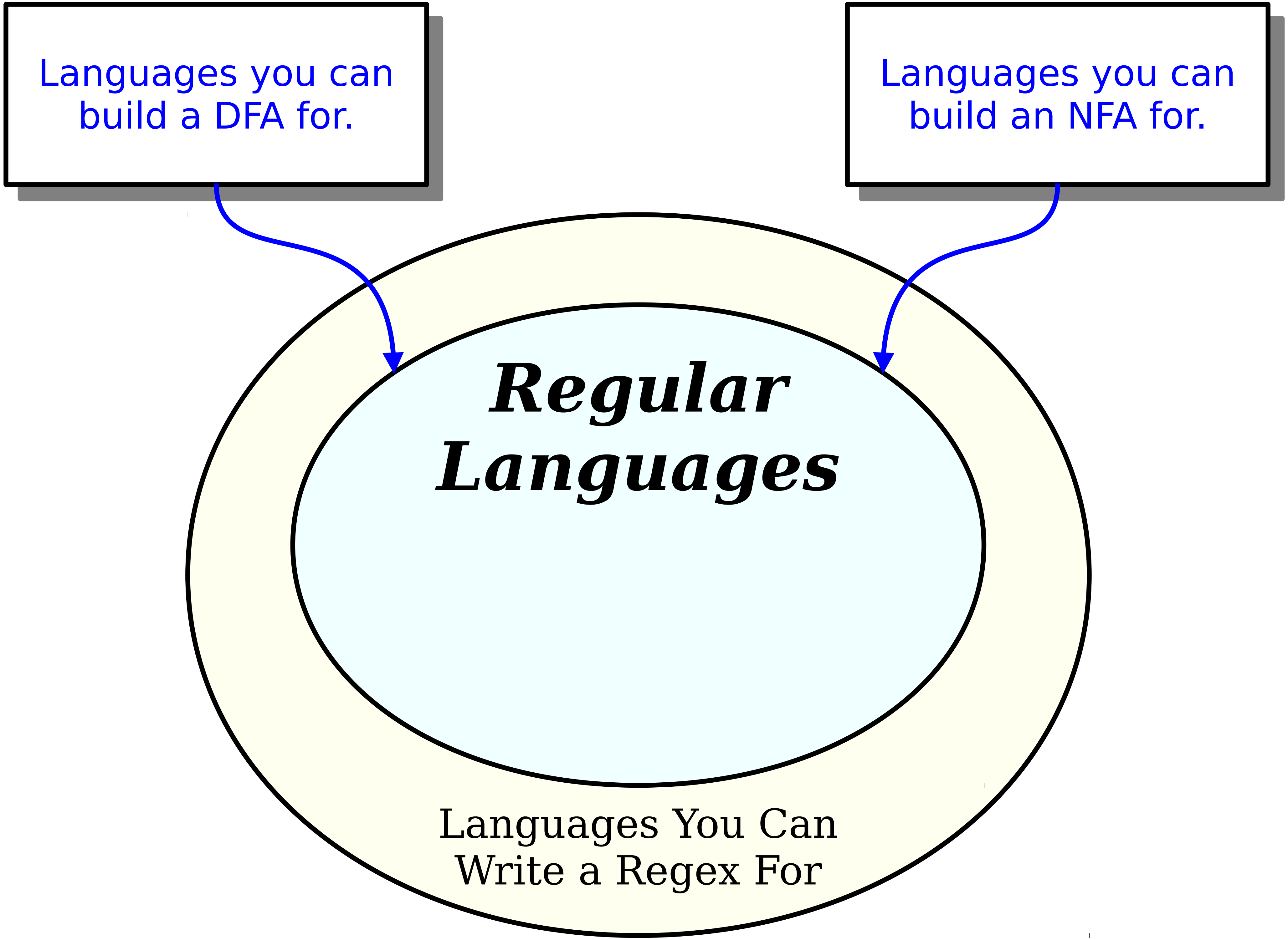

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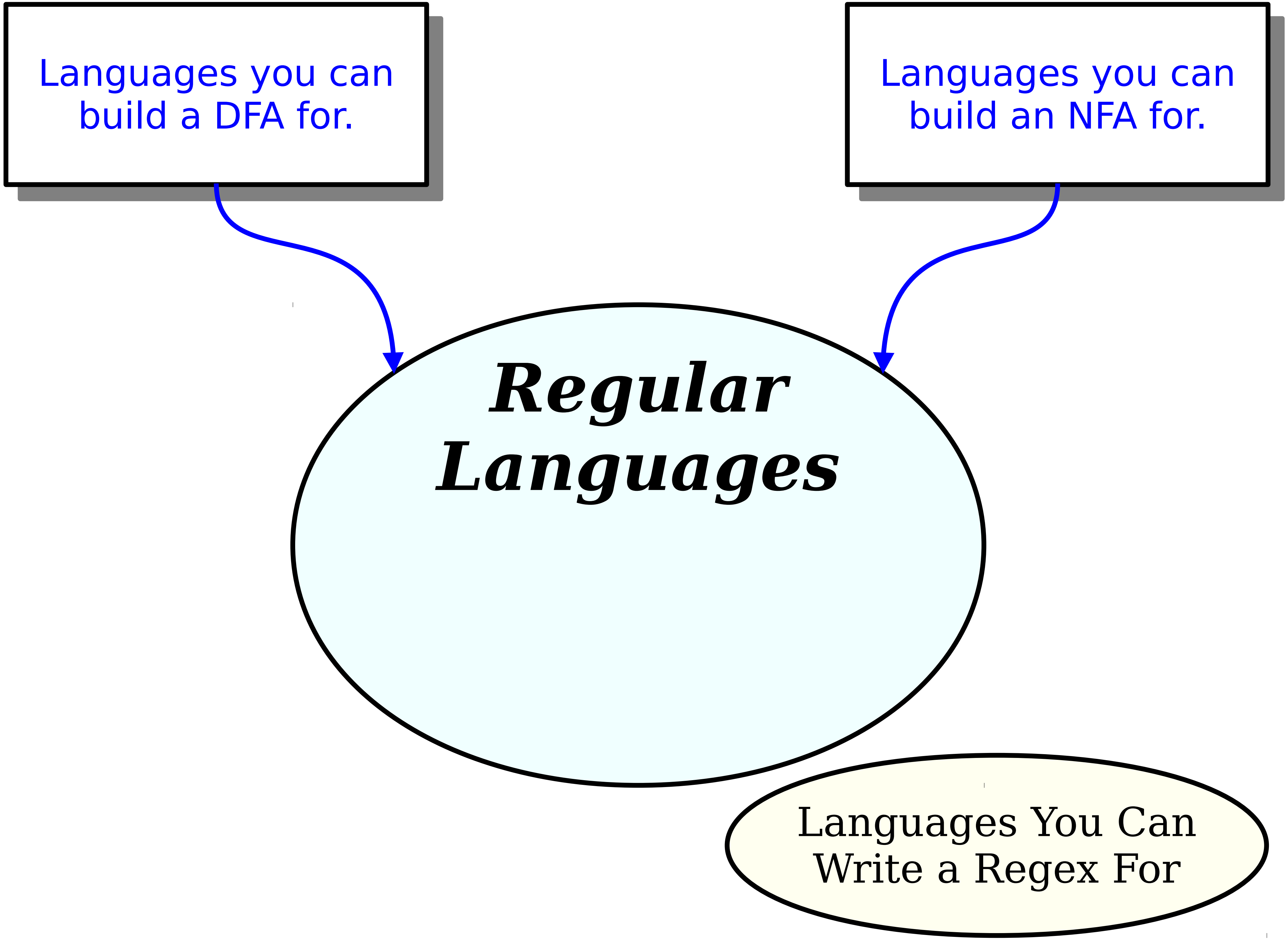
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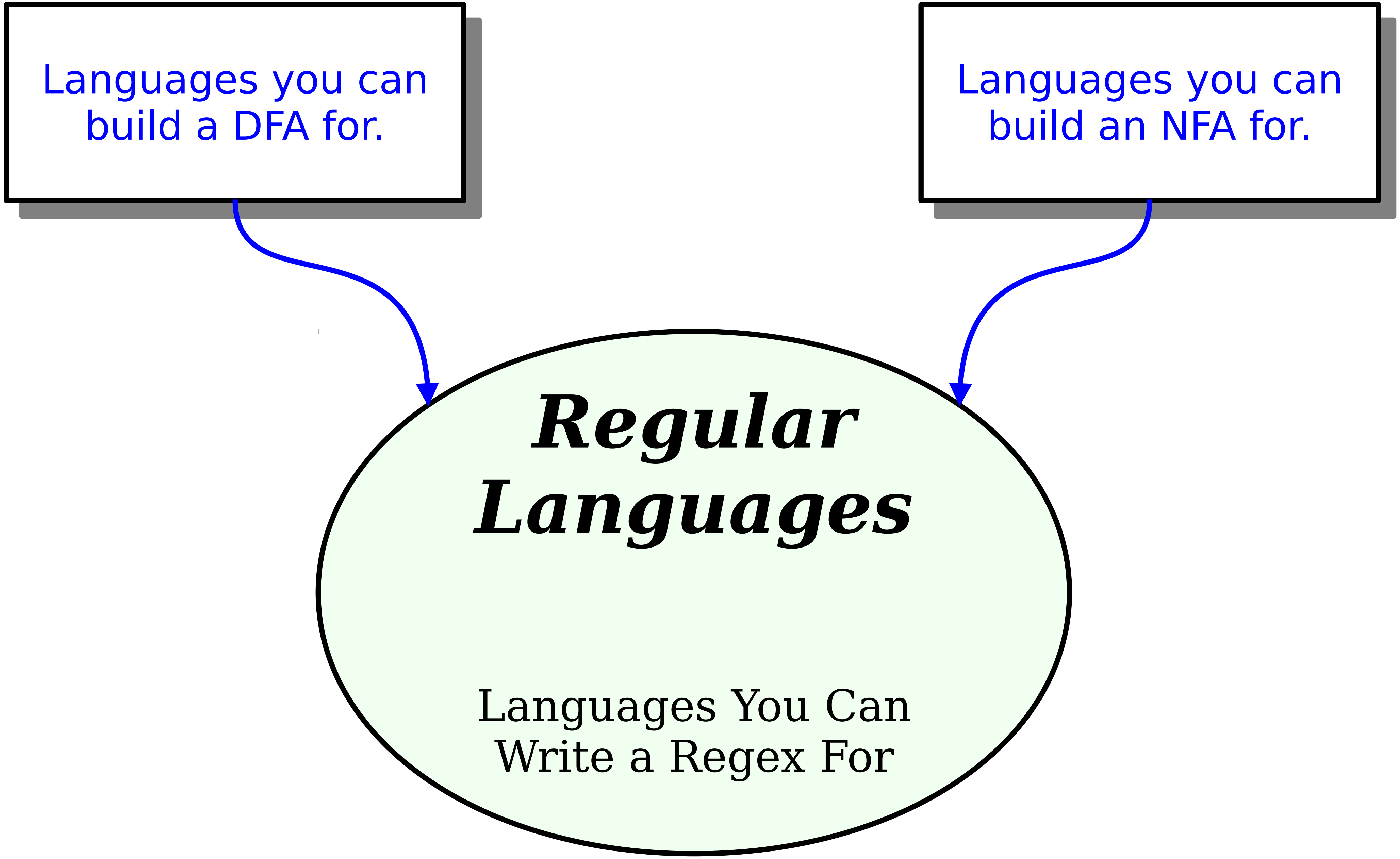
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Regular Languages
Languages You Can Write a Regex For

# The Power of Regular Expressions

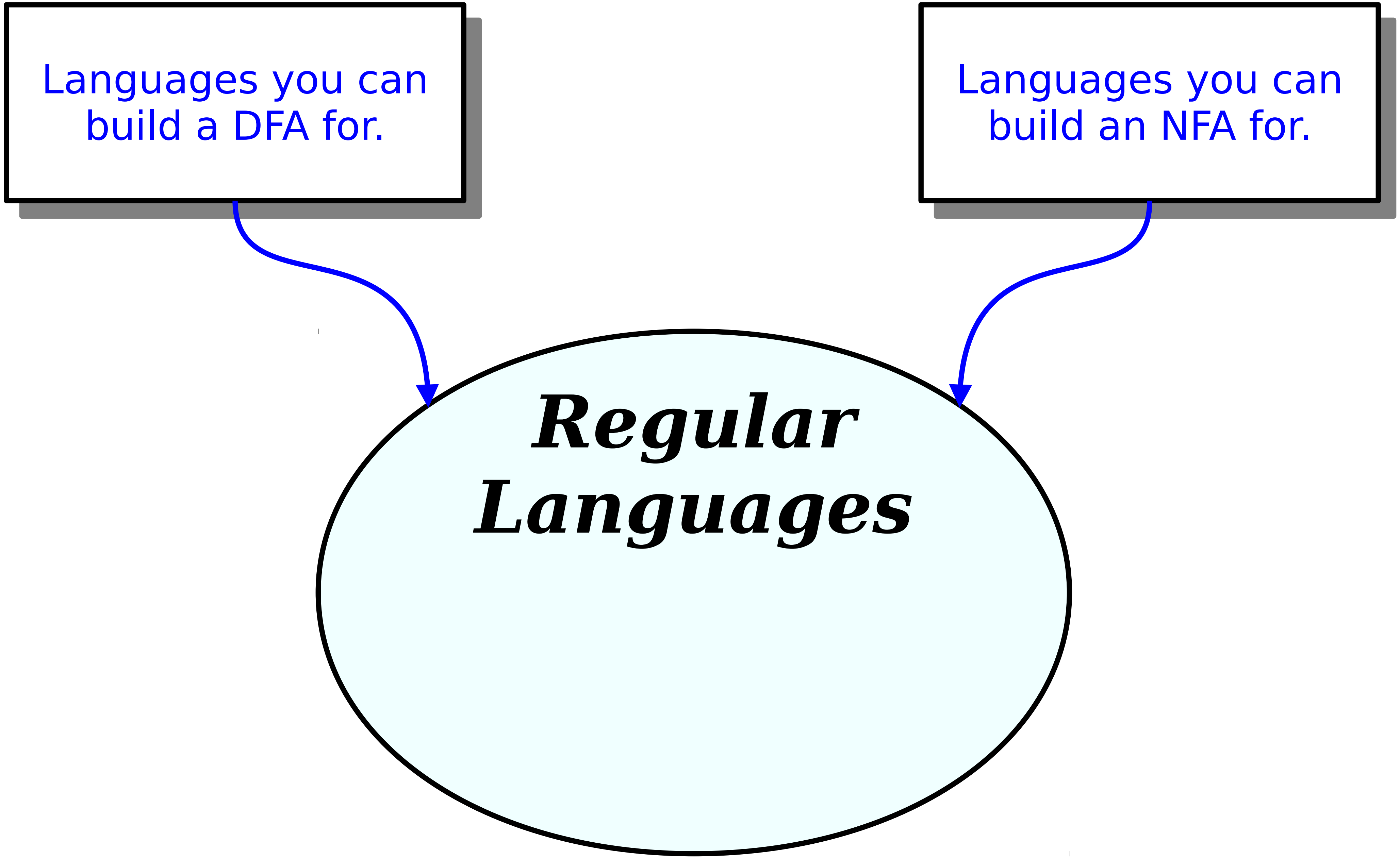
***Theorem:*** If $R$ is a regular expression, then $\mathscr{L}(R)$ is regular.

***Proof idea:*** Use induction!

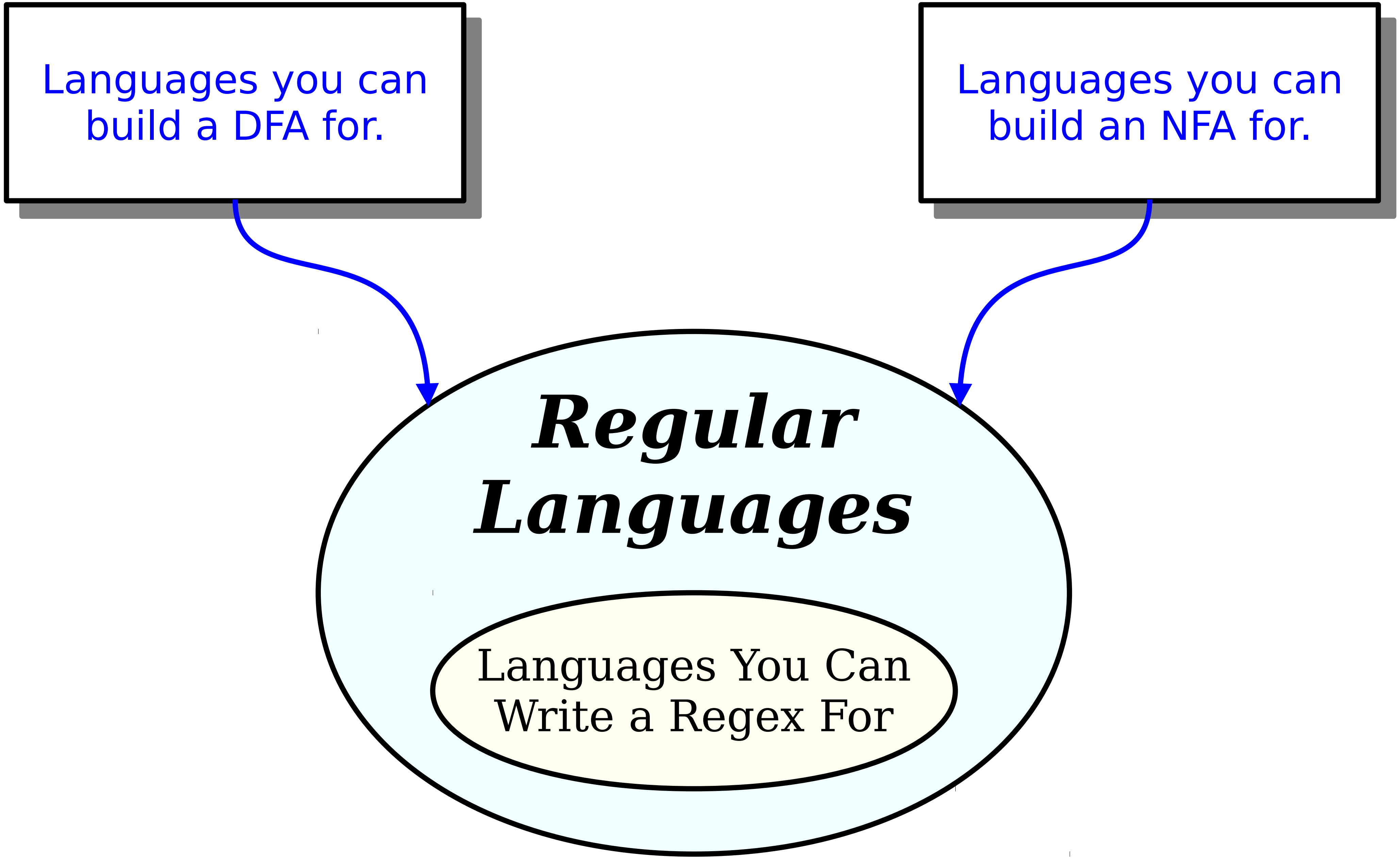
- The atomic regular expressions all represent regular languages.
- The combination steps represent closure properties.
- So anything you can make from them must be regular!

# Thompson’s Algorithm

- In practice, many regex matchers use an algorithm called ***Thompson's algorithm*** to convert regular expressions into NFAs (and, from there, to DFAs).
  - Read Sipser if you’re curious!
- ***Fun fact:*** the “Thompson” here is Ken Thompson, one of the co-inventors of Unix!



Languages you can build a DFA for.
Languages you can build an NFA for.
Regular Languages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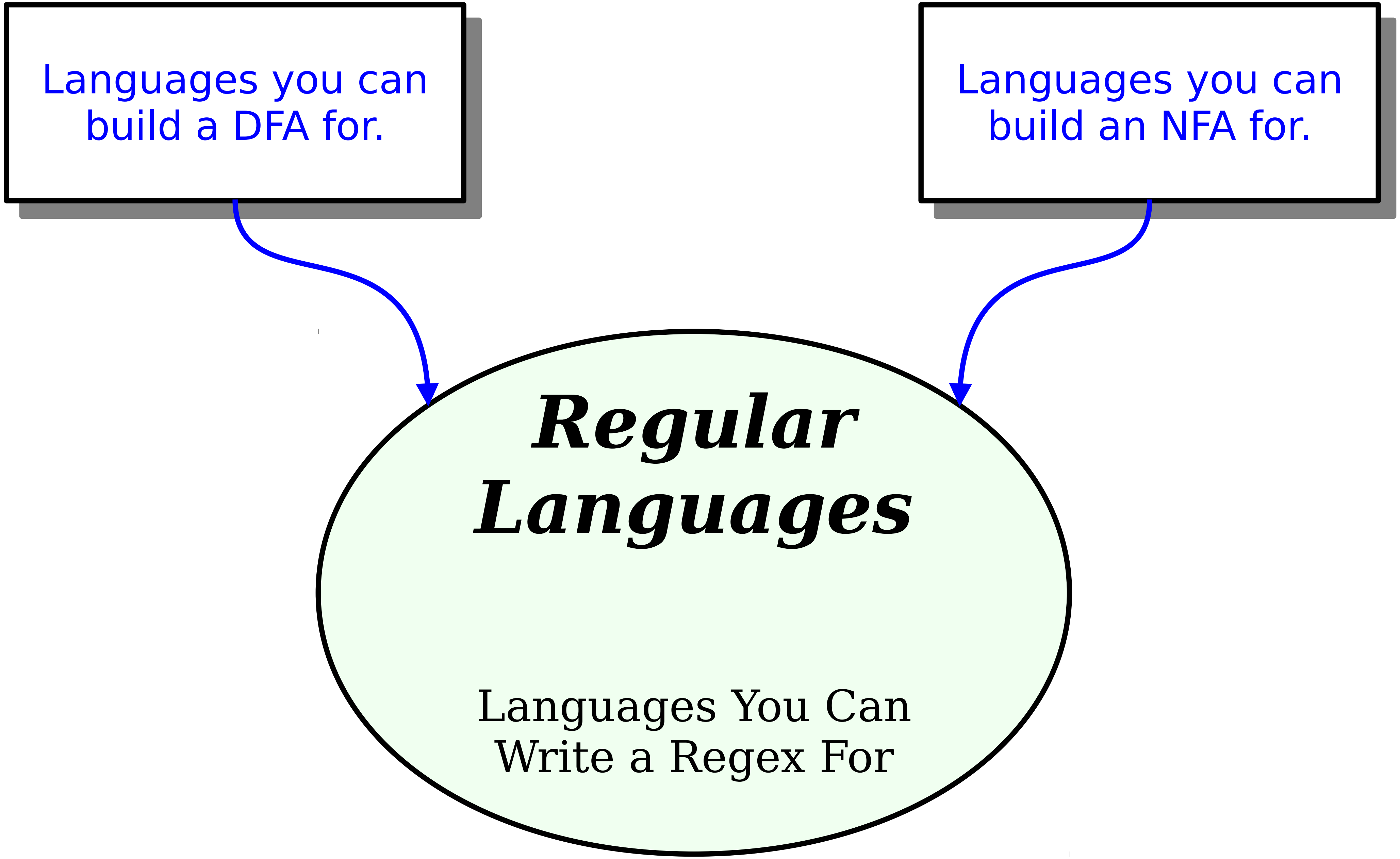
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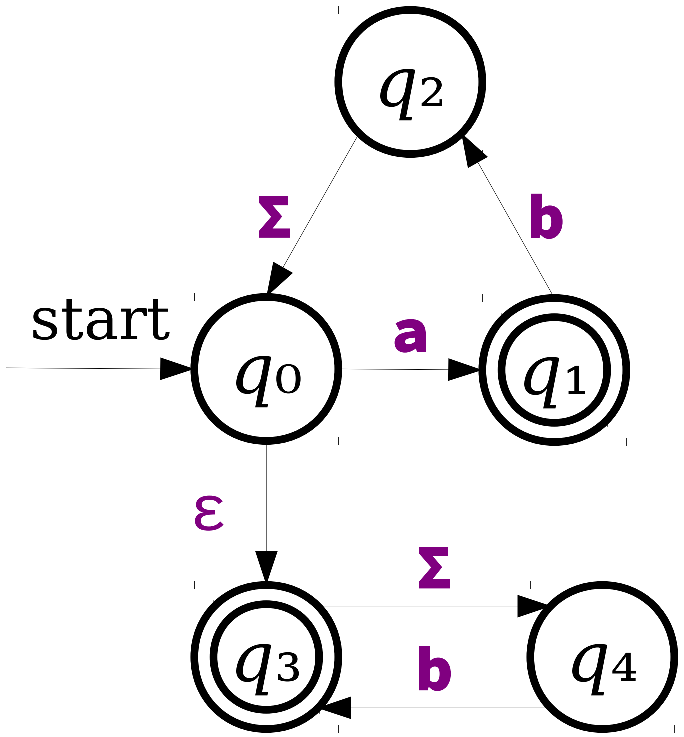
# The Power of Regular Expressions

***Theorem:*** If $L$ is a regular language, then there is a regular expression for $L$.

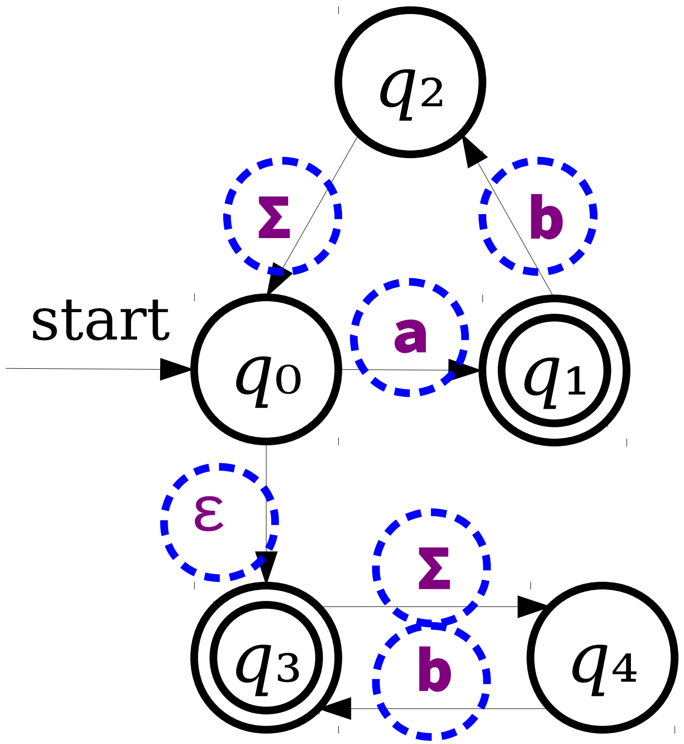
***This is not obvious!***

***Proof idea:*** Show how to convert an arbitrary NFA into a regular expression.

# Generalizing NFAs



# Generalizing NFAs



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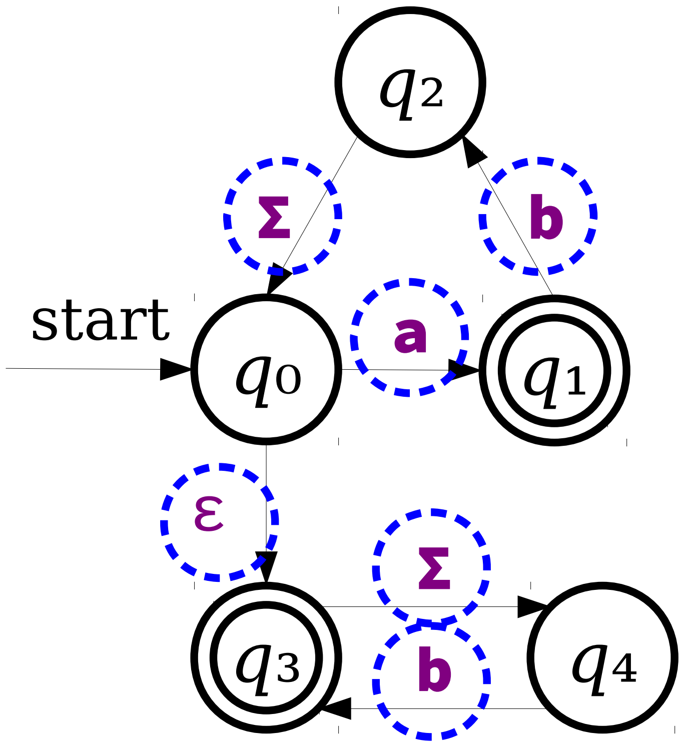
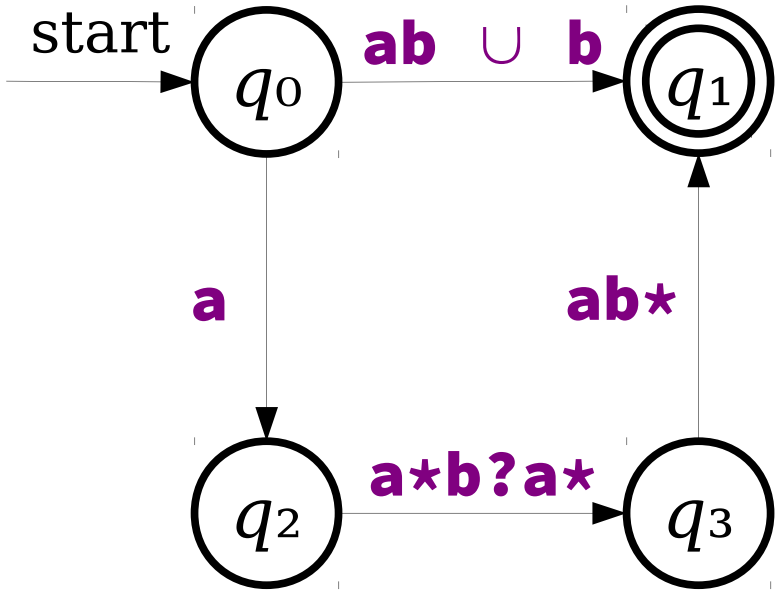



These are all regular expressions!

# Generalizing NFAs



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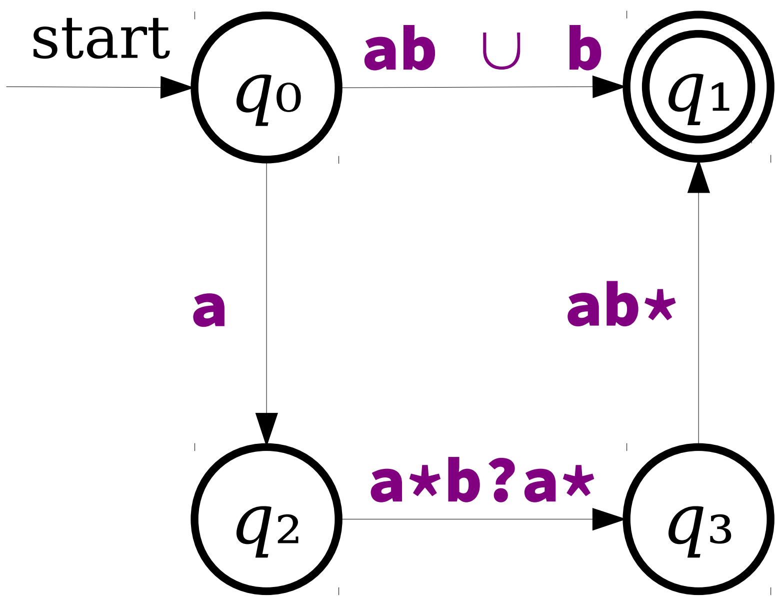
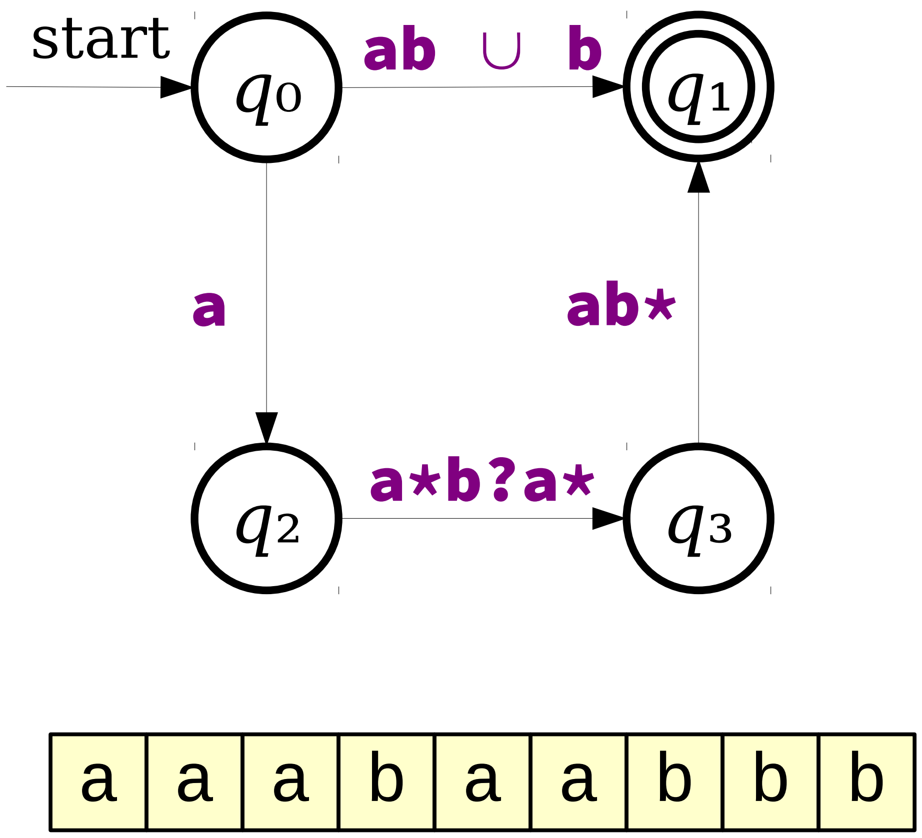



Note: Actual NFAs aren't allowed to have transitions like these. This is just a thought experiment.

# Generalizing NFAs



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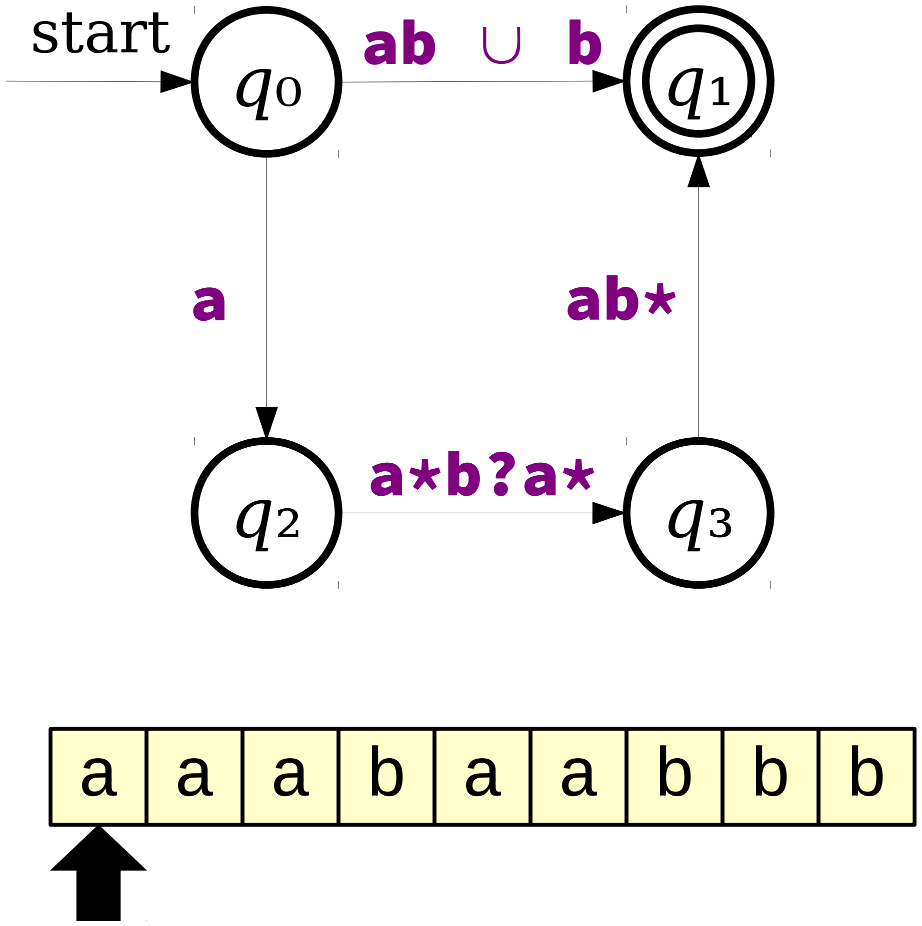

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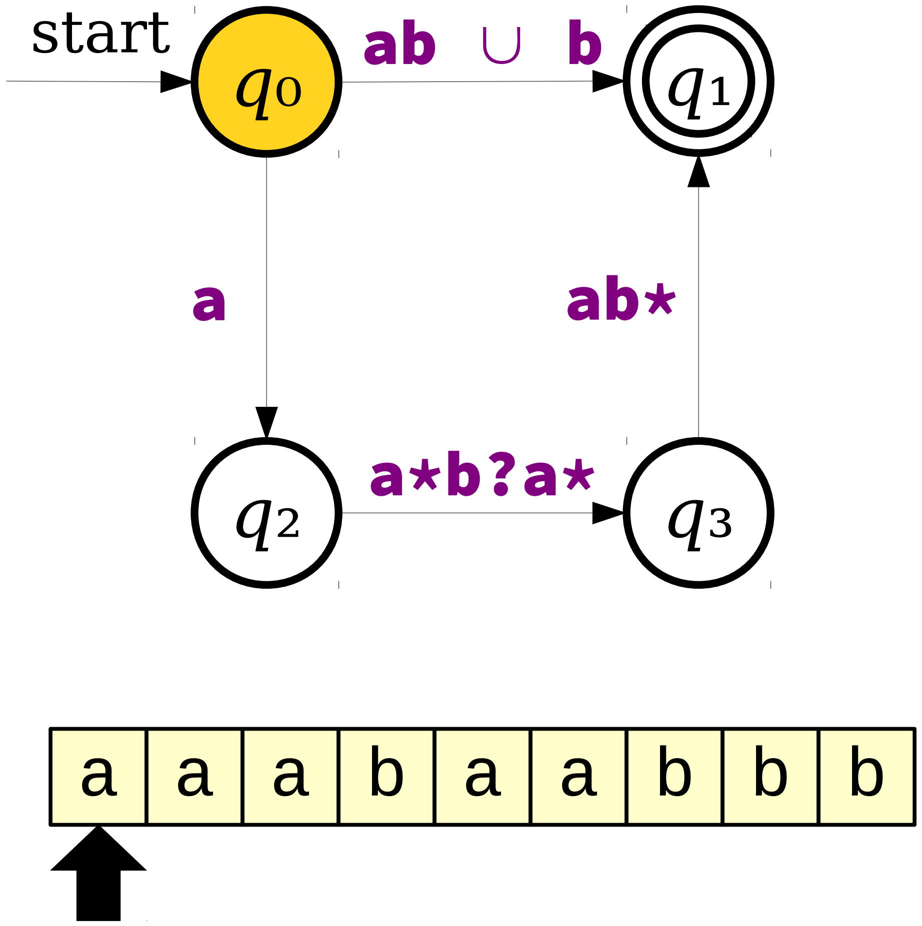

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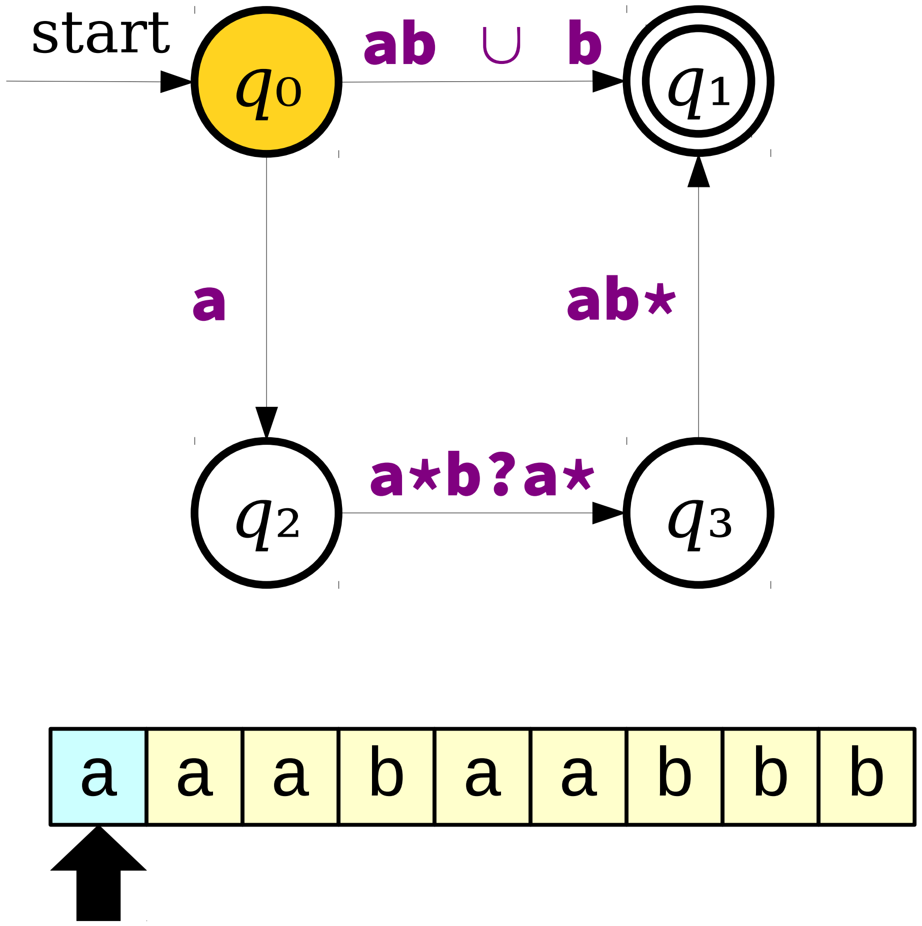

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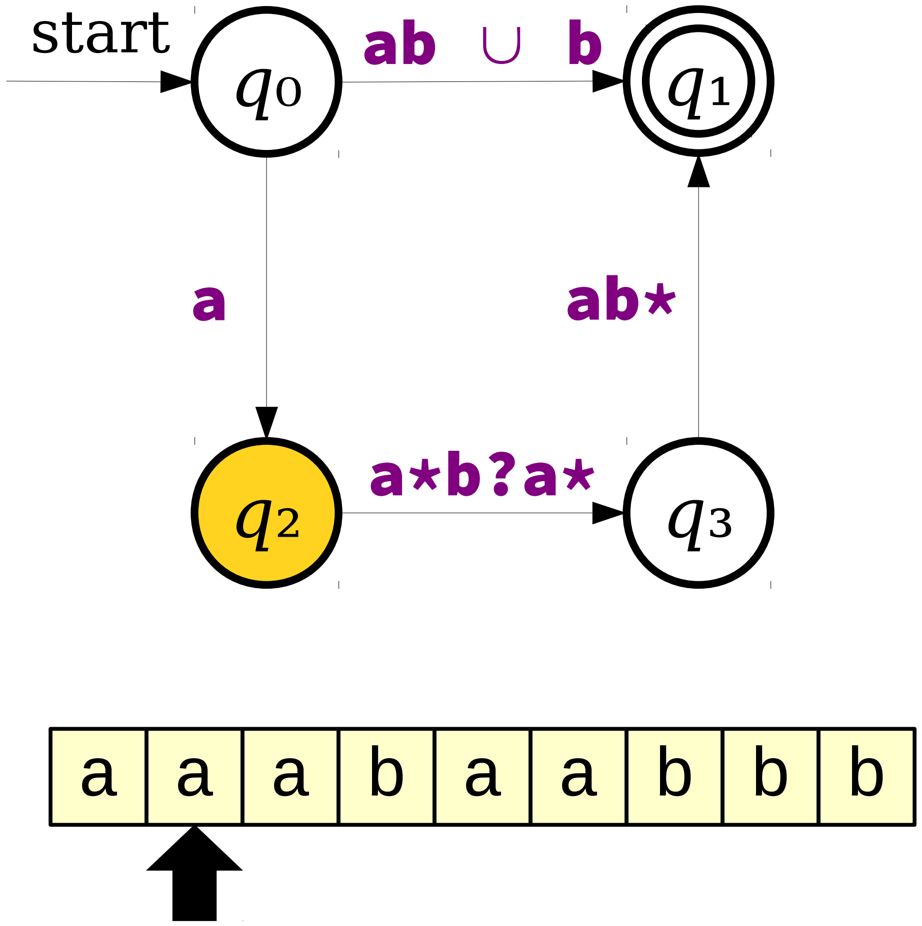

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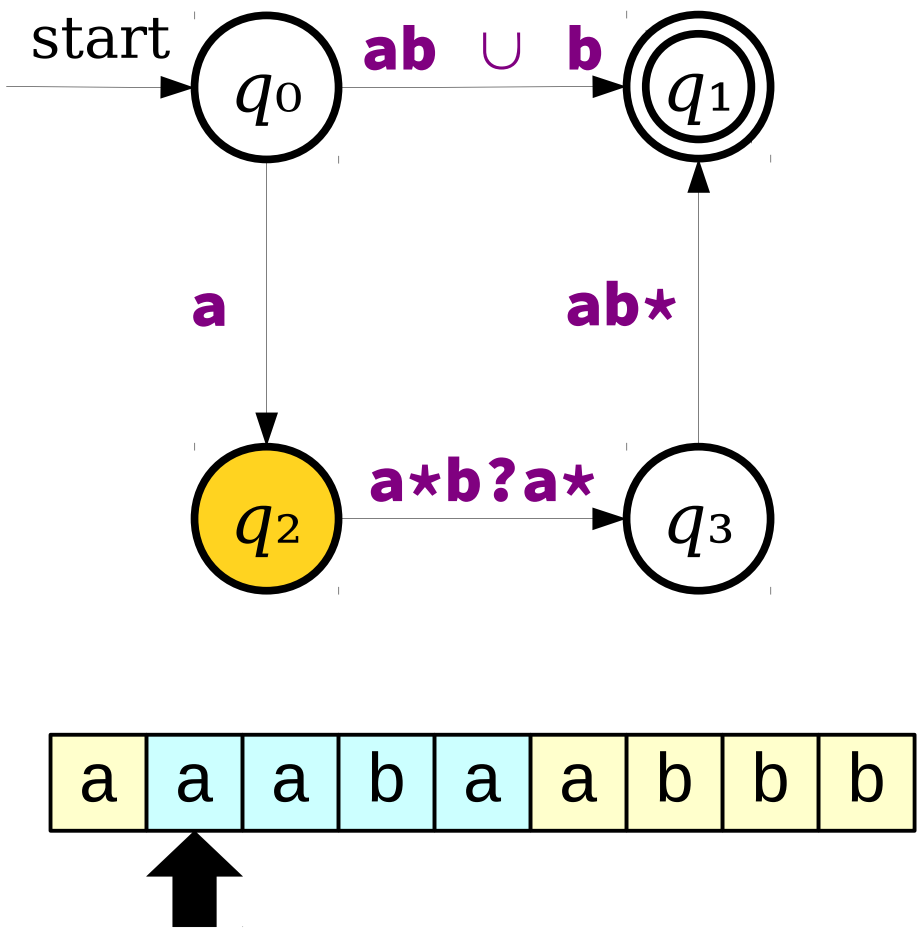

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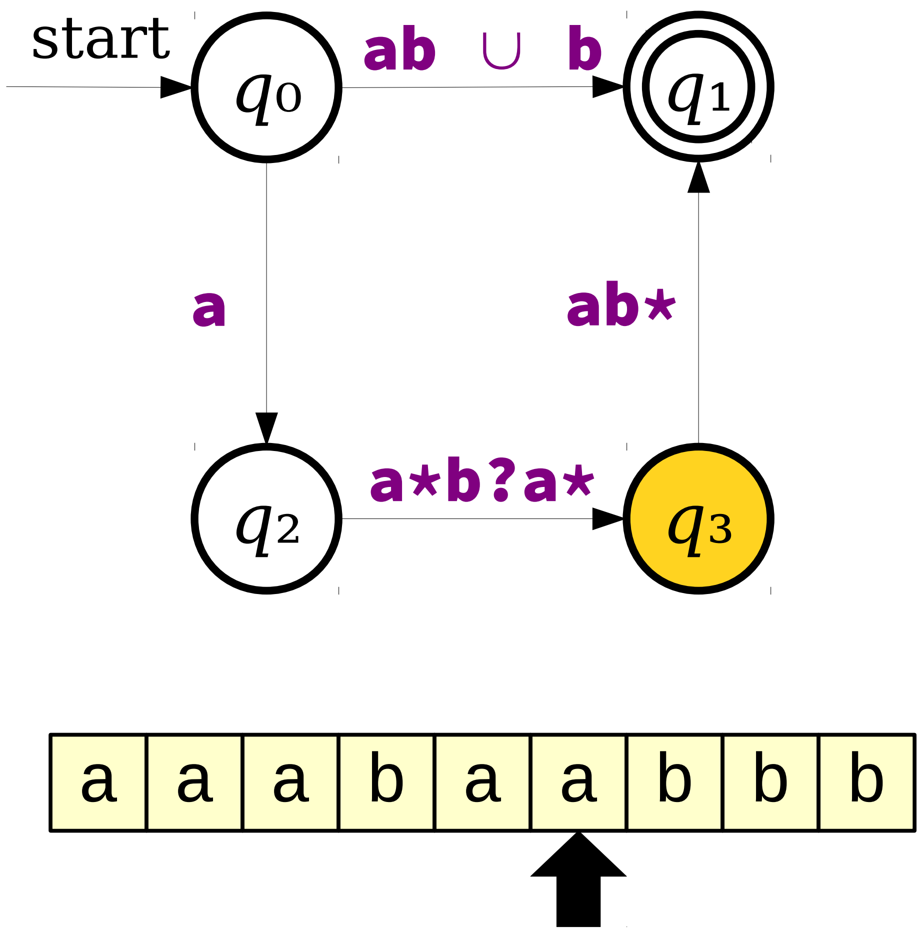

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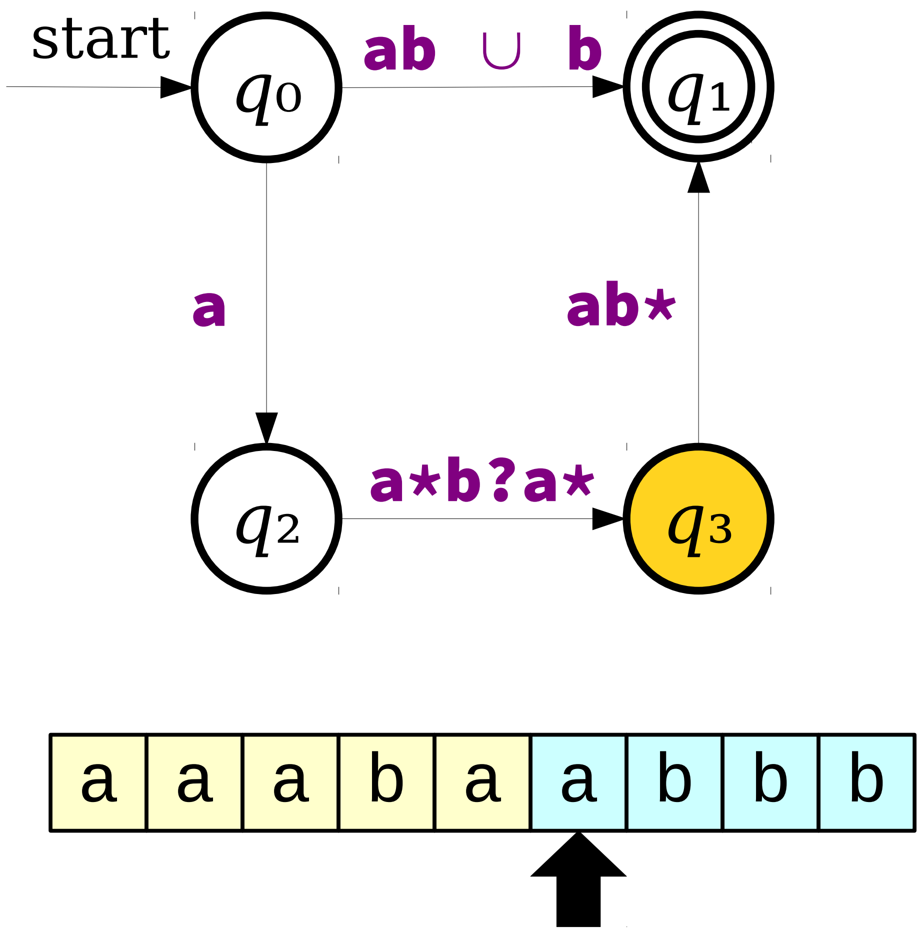
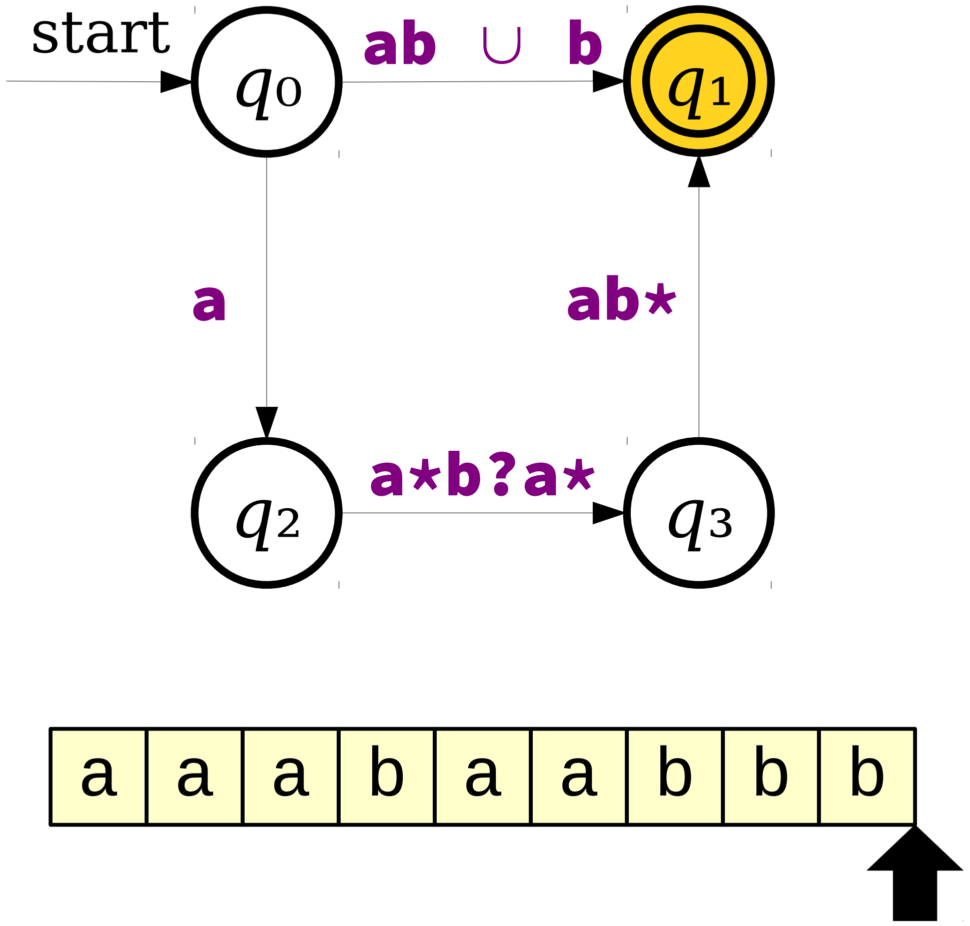


# Generalizing NFAs



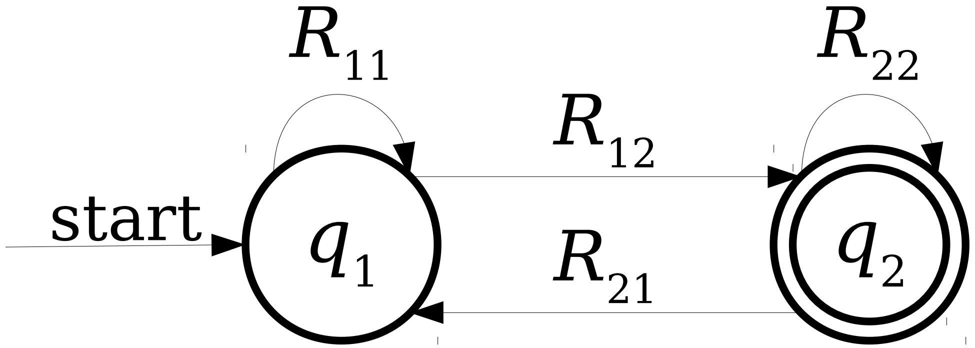
***Key Idea 1:*** Imagine that we can label transitions in an NFA with arbitrary regular expressions.

# Generalizing NFAs

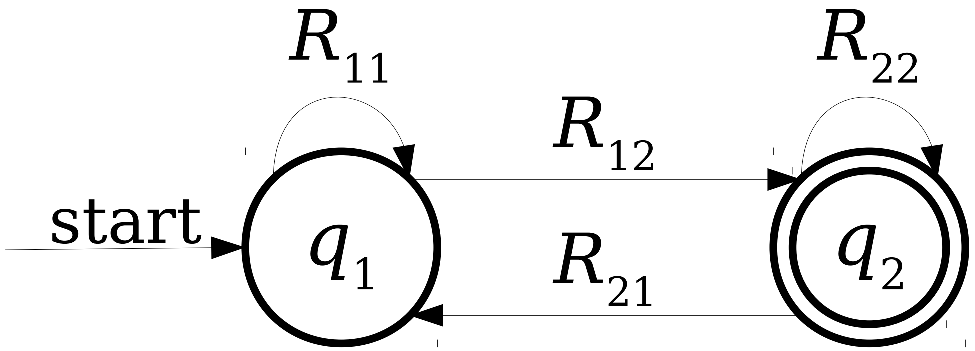
***Key Idea 2:*** If we can convert an NFA into a generalized NFA that looks like this…


…then we can easily read off a regular expression for the original NFA.

# From NFAs to Regular Expressions

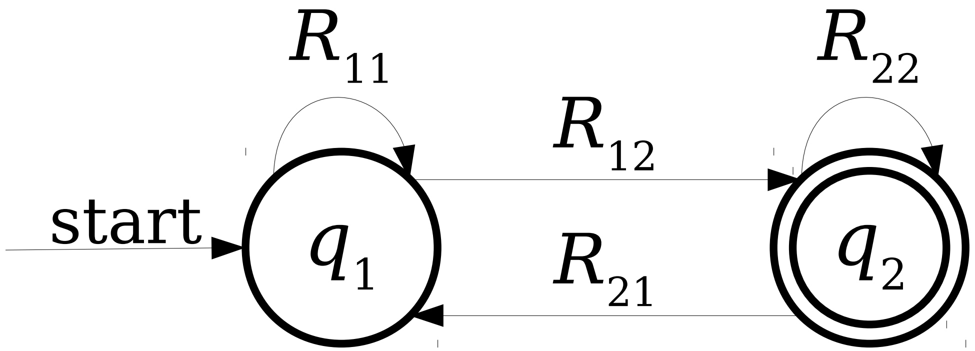


# From NFAs to Regular Expressions



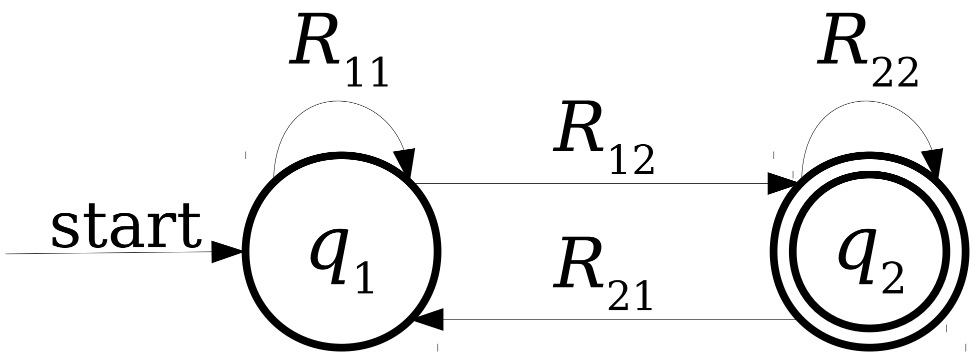

Here, $R_{11}$, $R_{12}$, $R_{21}$, and $R_{22}$ are arbitrary regular expressions.

# From NFAs to Regular Expressions



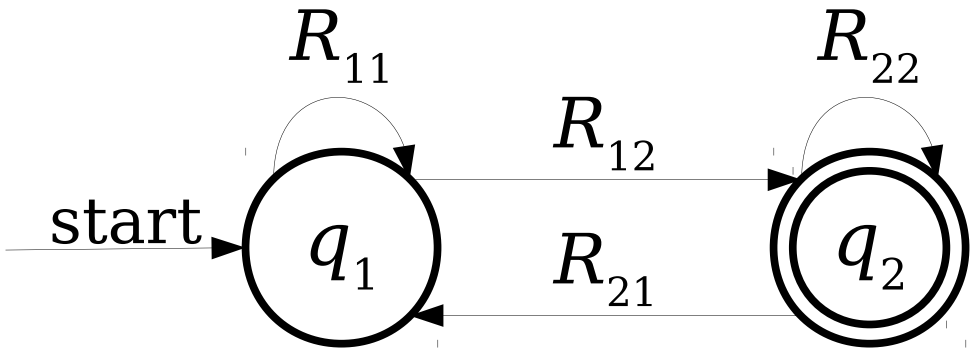

Question: Can we get a clean regular expression from this NFA?

# From NFAs to Regular Expressions



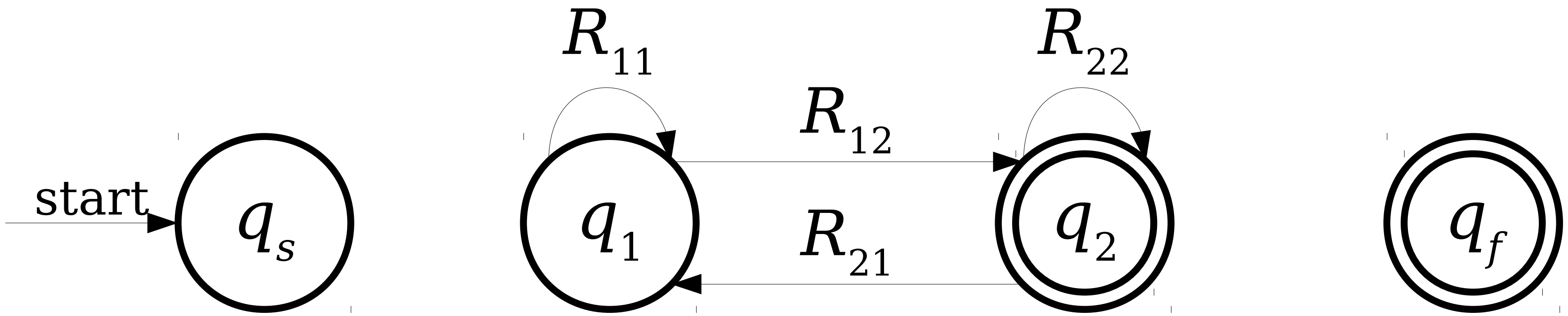

**Key Idea 3:** Somehow transform this NFA so that it looks like this:

# From NFAs to Regular Expressions

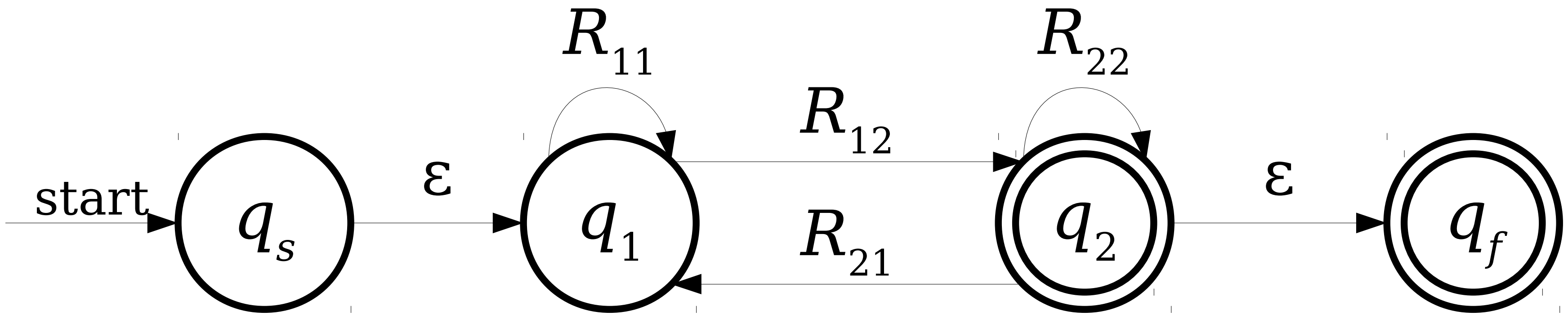



The first step is going to be a bit weird...

# From NFAs to Regular Expressions



# From NFAs to Regular Expressions



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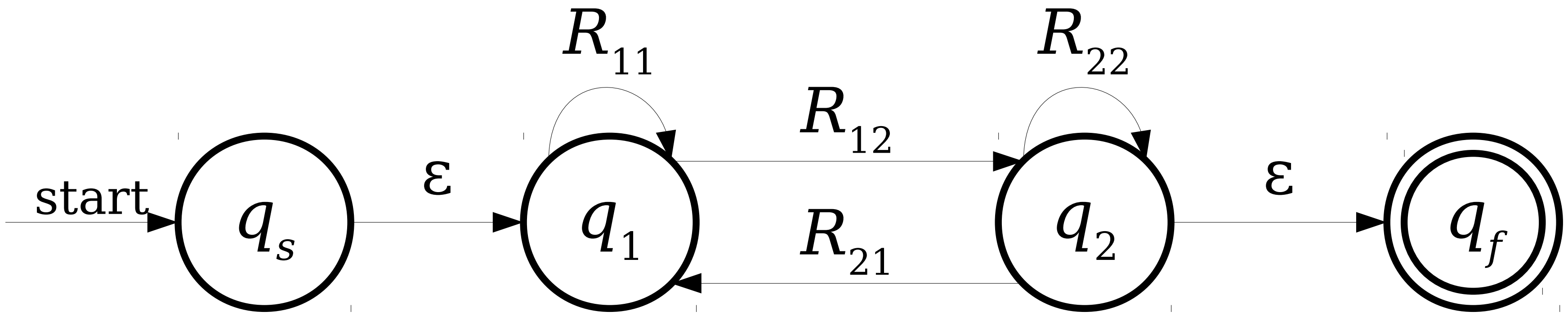

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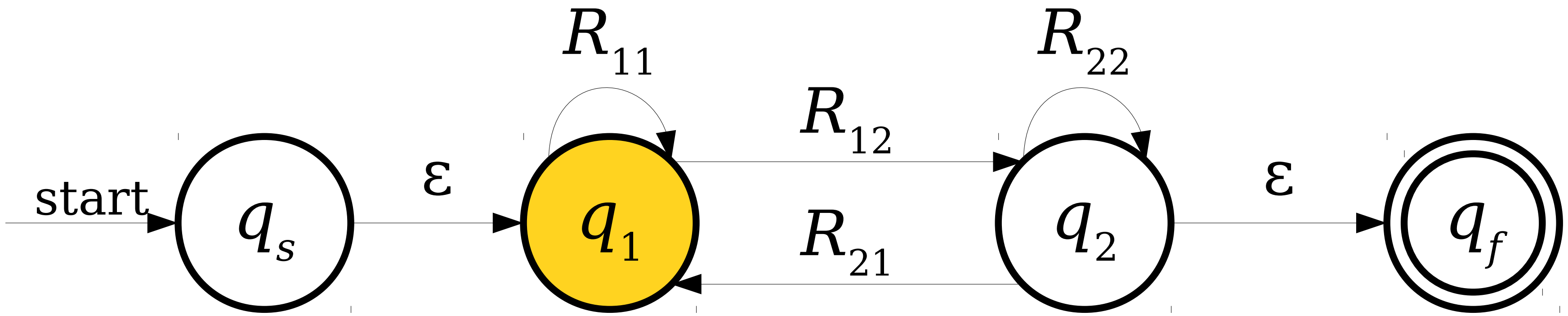

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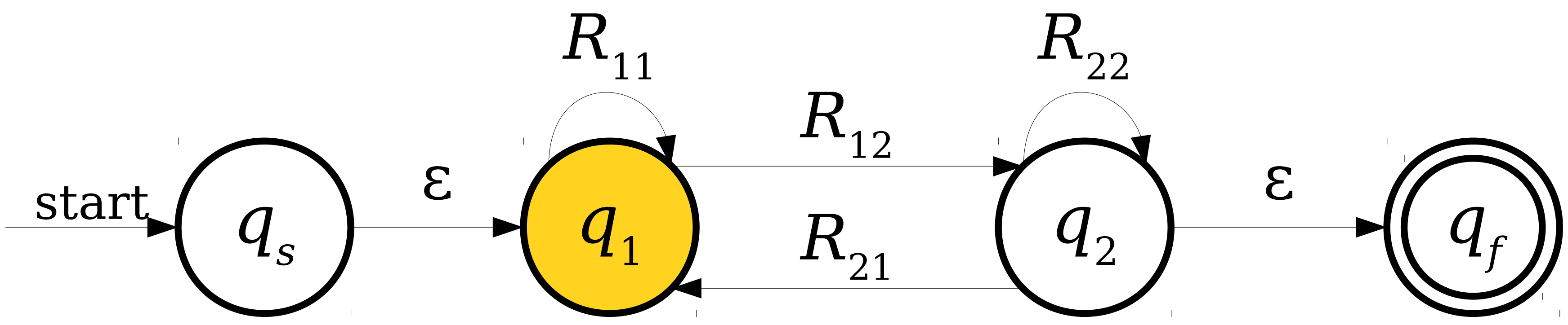
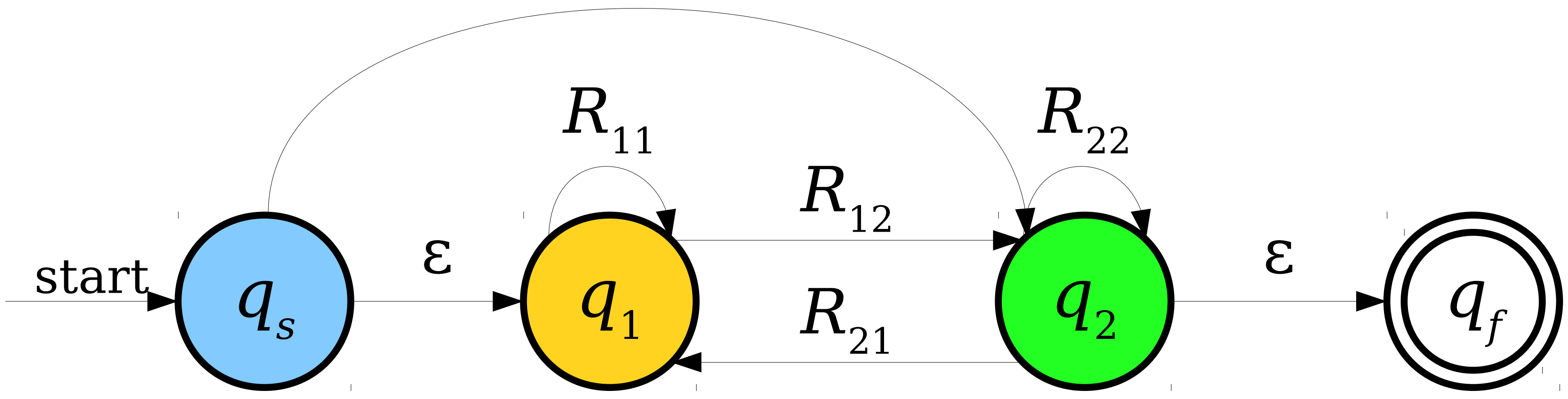



Could we eliminate this state from the NFA?

# From NFAs to Regular Expressions



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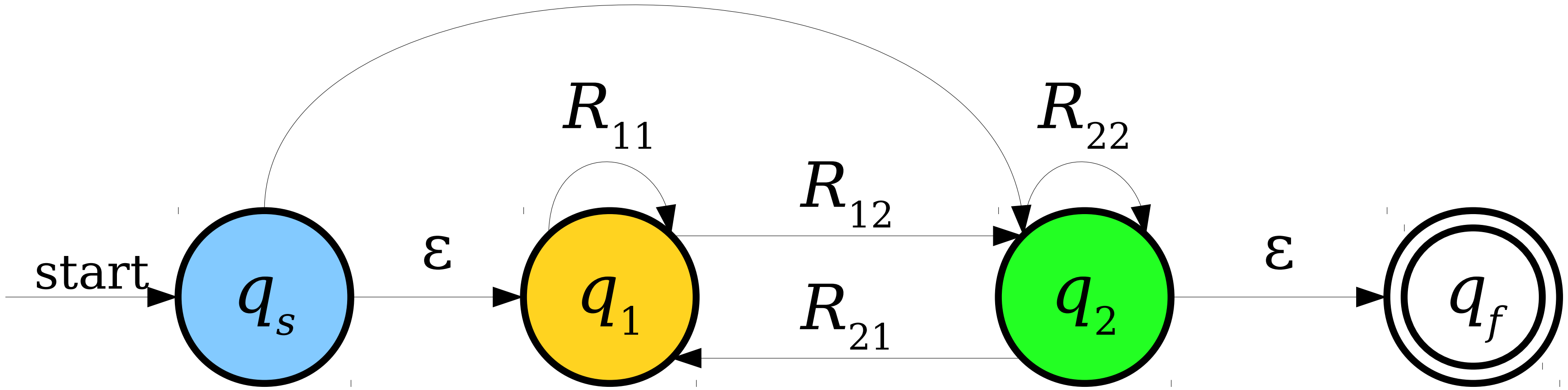
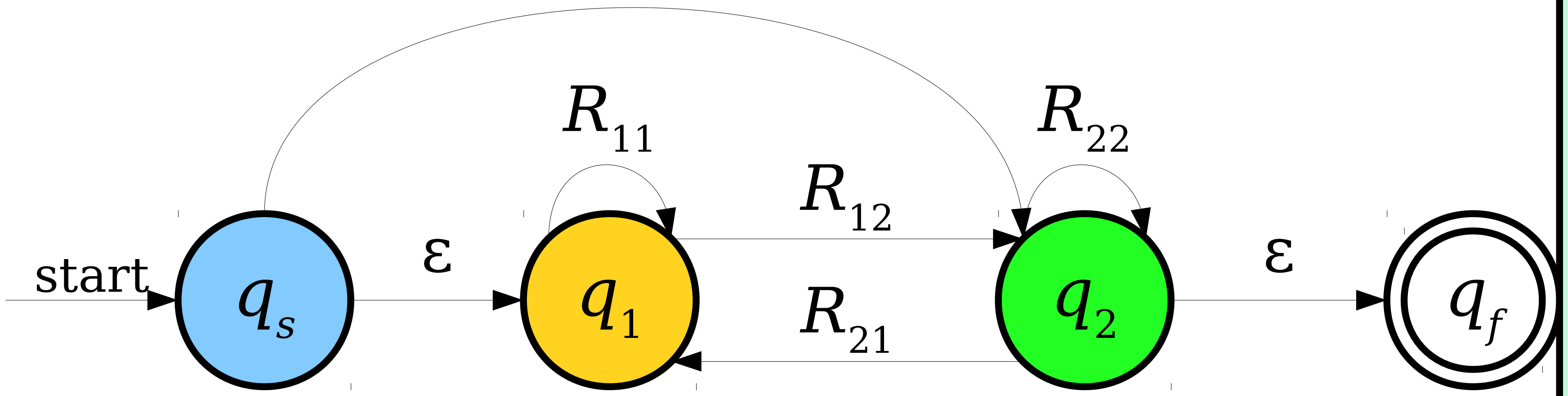



Here is a pattern that we might process when going from $q_s$ to $q_2$: $\varepsilon R_{12}$

# From NFAs to Regular Expressions




Here is a pattern that we might process when going from $q_s$ to $q_2$: $\varepsilon R_{12}$

**State elimination quick check:**

How many of the following are also patterns might we process when going from $q_s$ to $q_2$?

$\varepsilon R_{11} R_{12}$

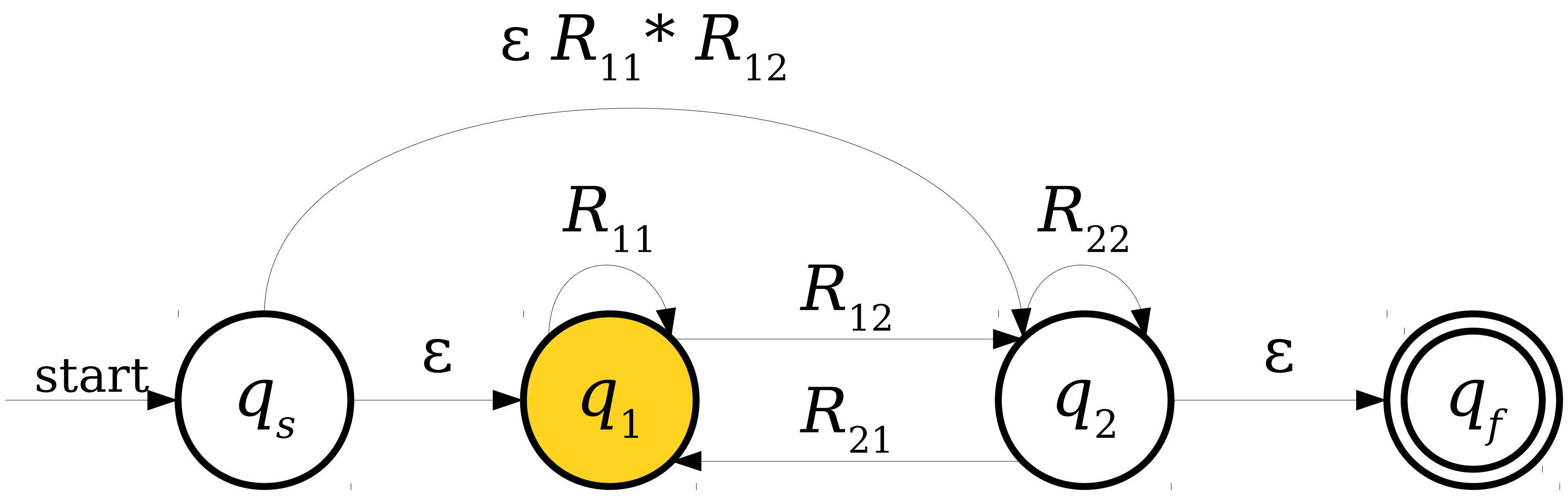
$\varepsilon R_{11} R_{11} R_{12}$

$\varepsilon R_{11} R_{12} R_{11}$

$\varepsilon R_{11} R_{12} R_{21}$

$\varepsilon R_{11} R_{12} R_{21} R_{11} R_{12}$

# From NFAs to Regular Expressions



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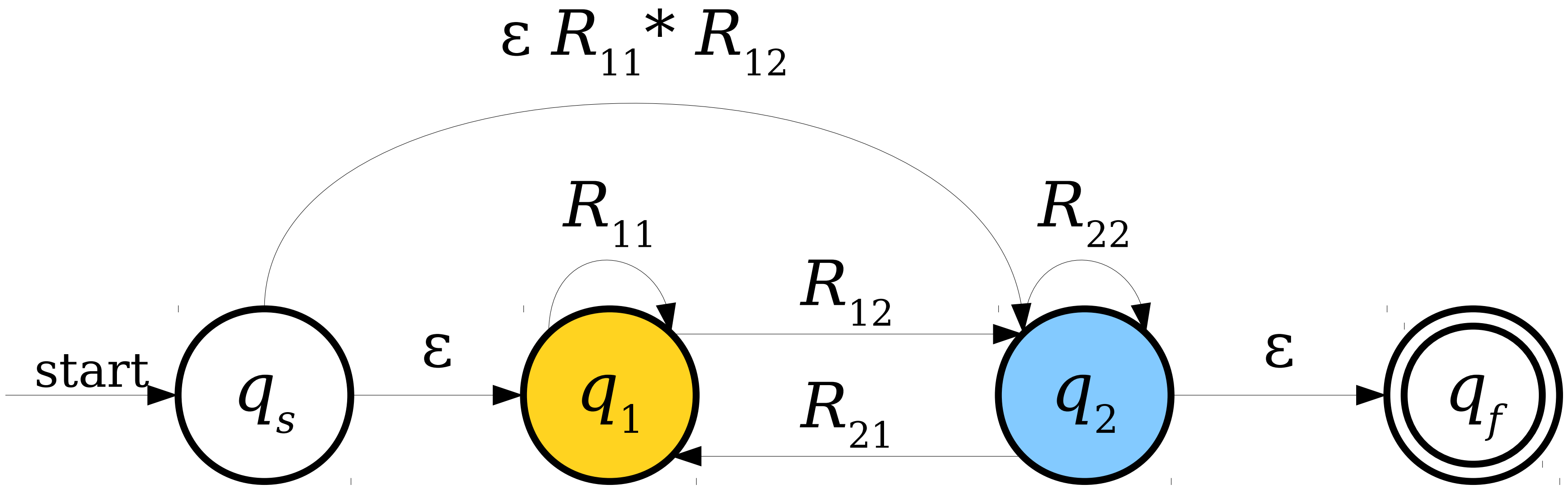

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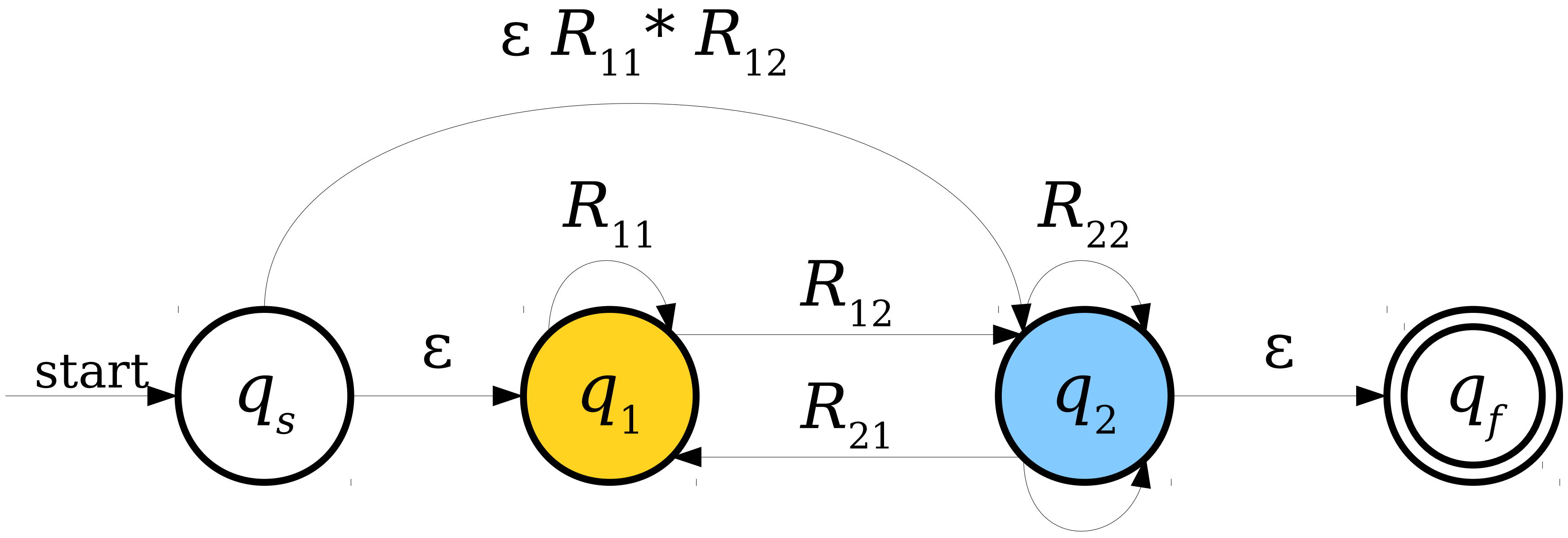

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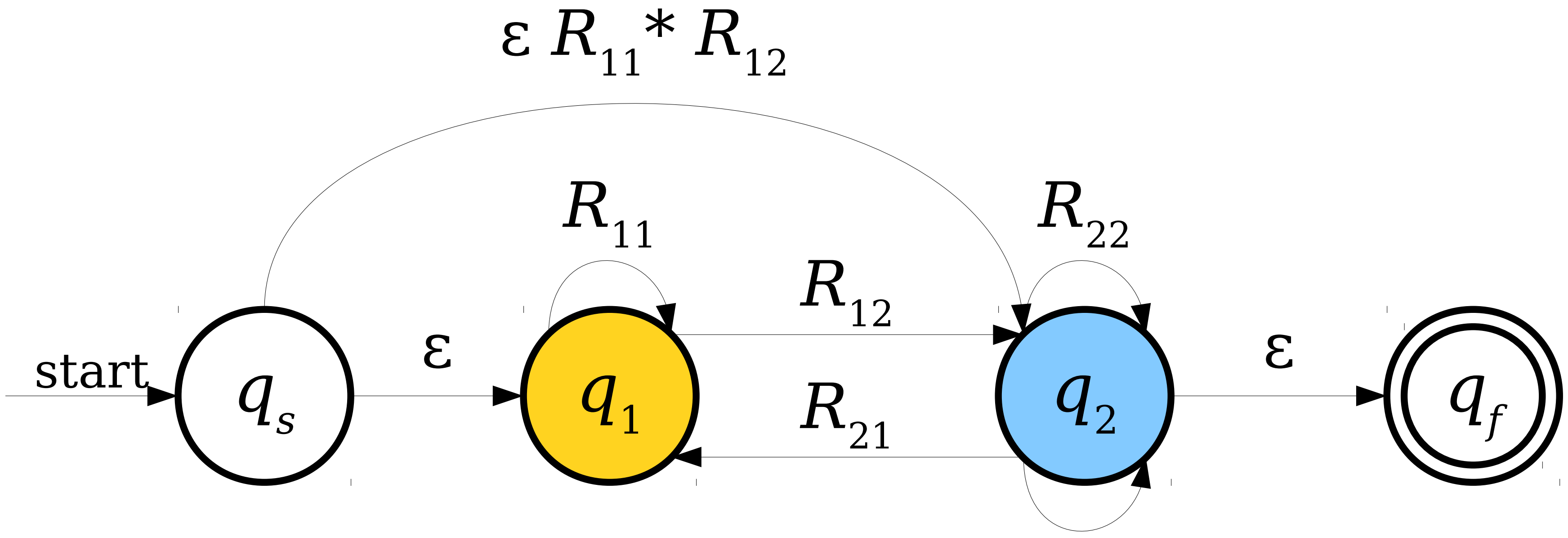
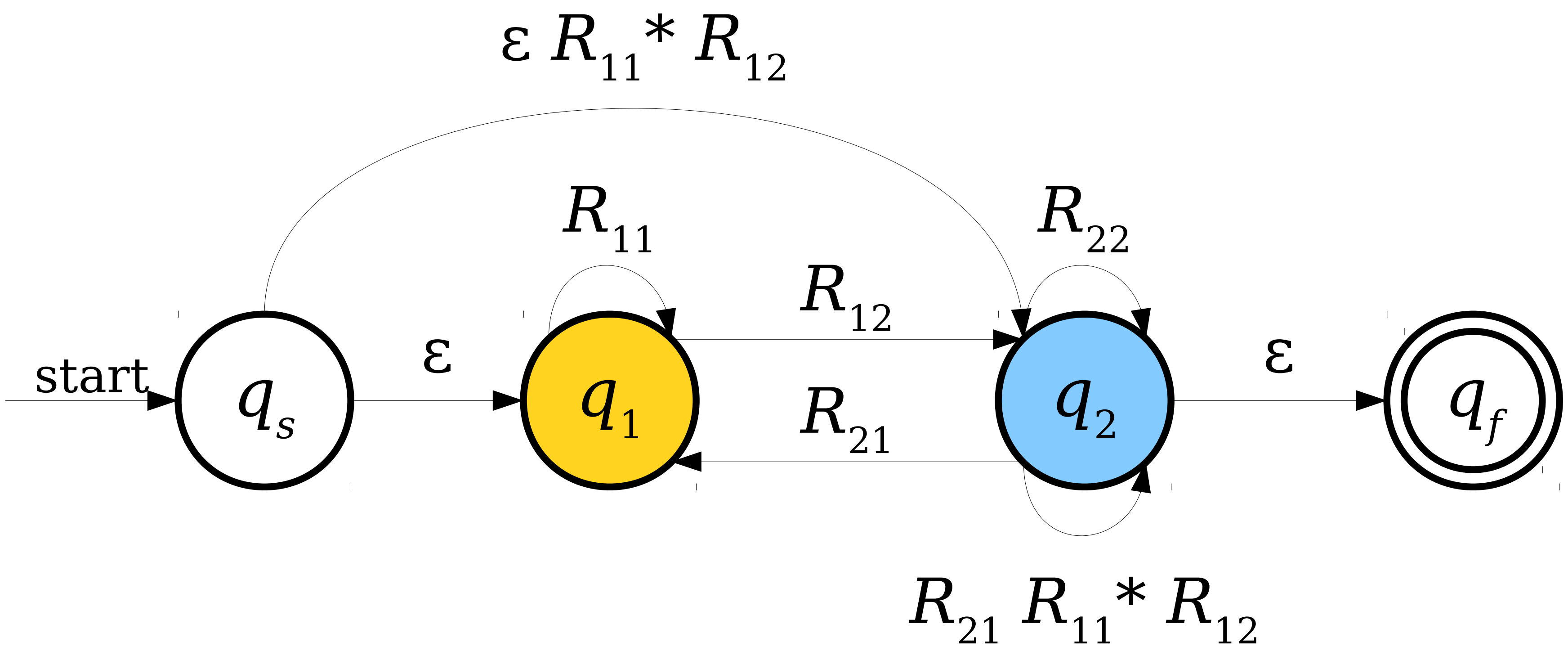



Here is a pattern that we might process when going from $q_2$ to $q_2$: $R_{21}R_{11}R_{12}$

# From NFAs to Regular Expressions




Here is a pattern that we might process when going from $q_2$ to $q_2$: $R_{21}R_{11}R_{12}$

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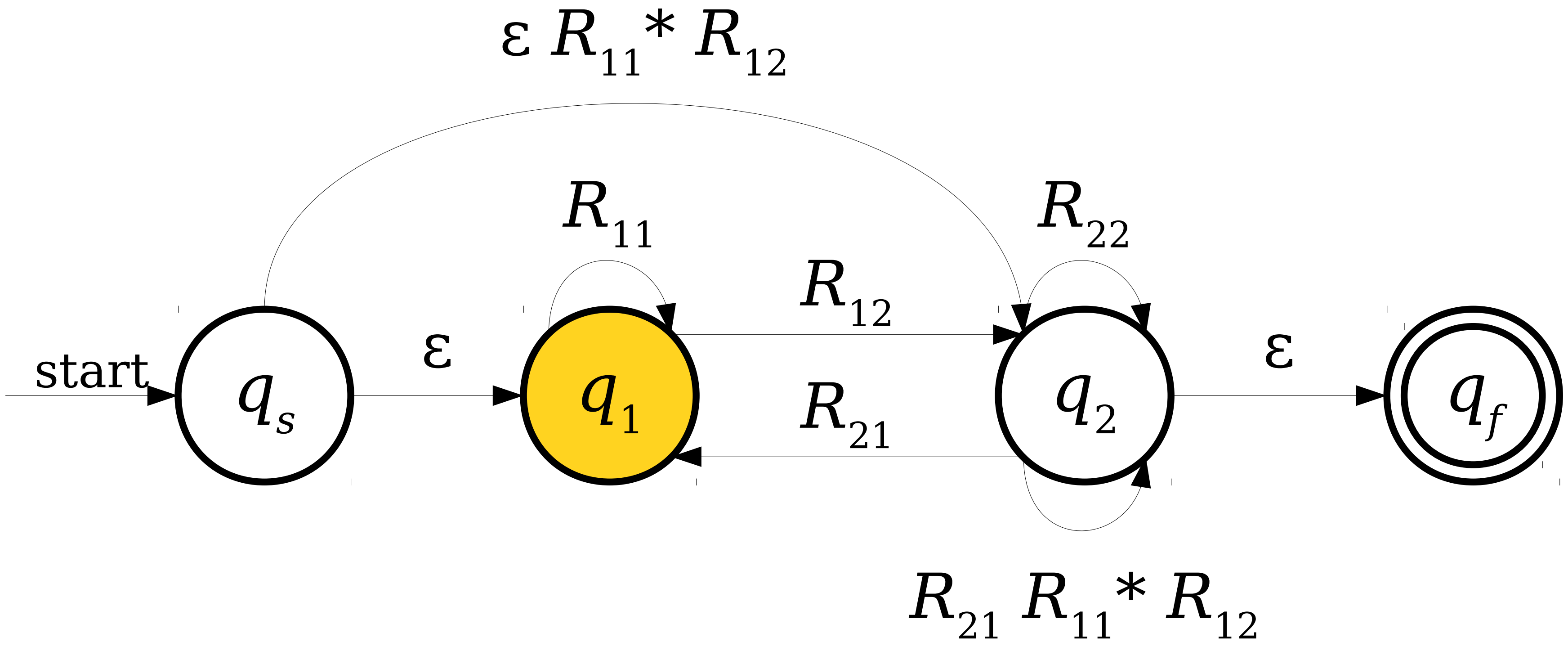

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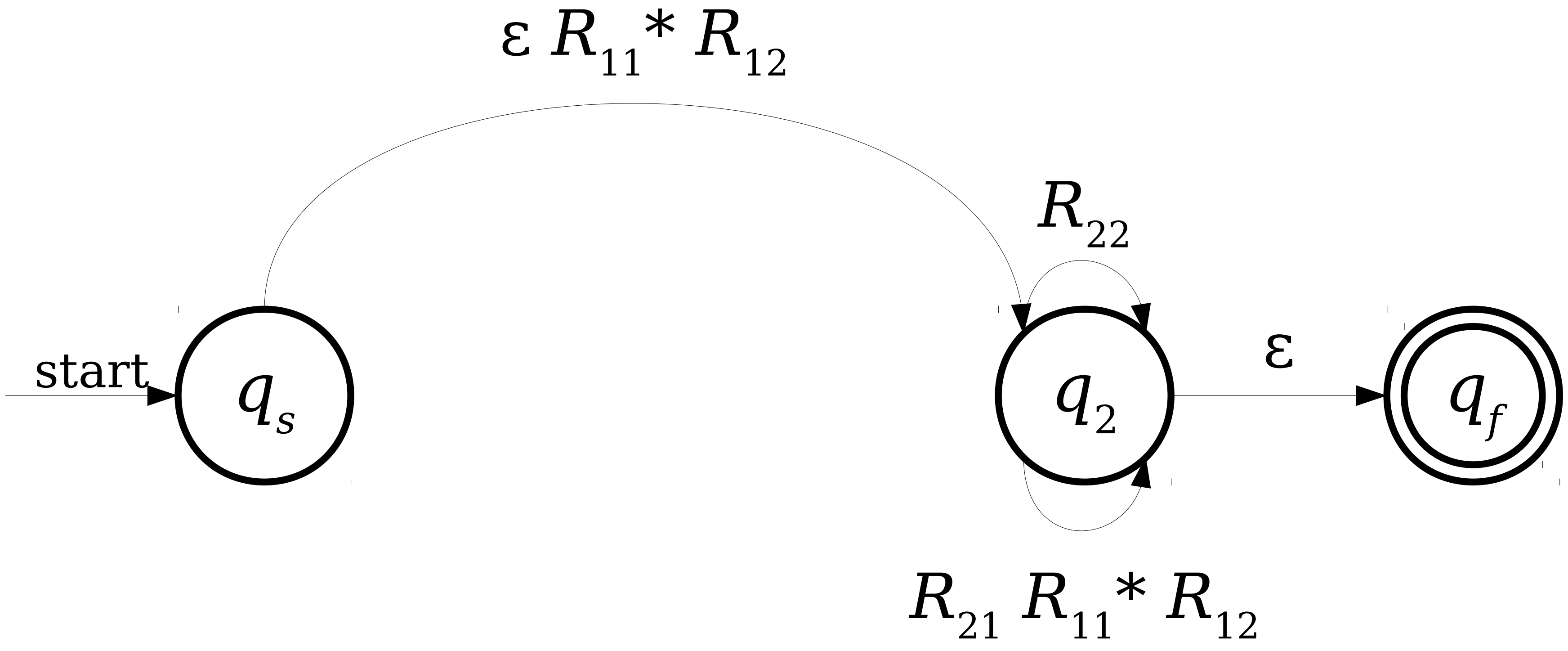

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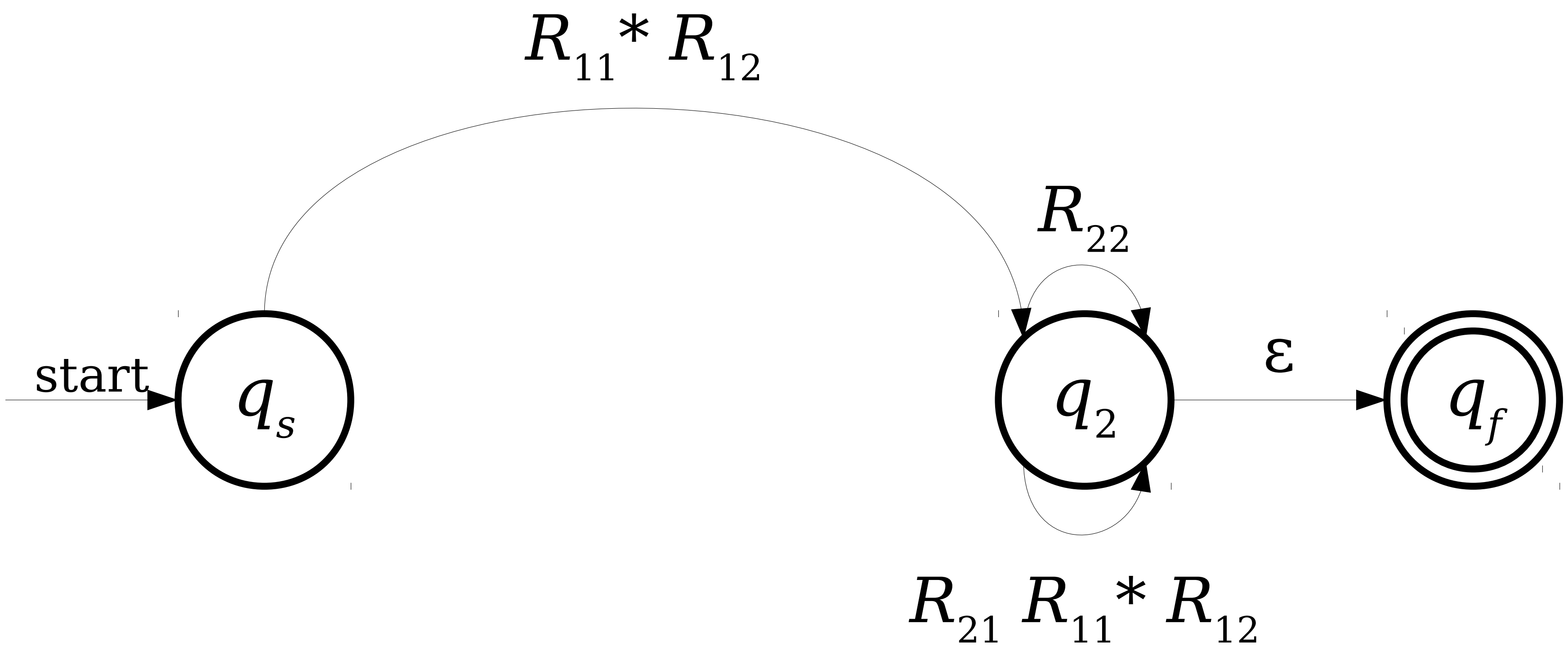

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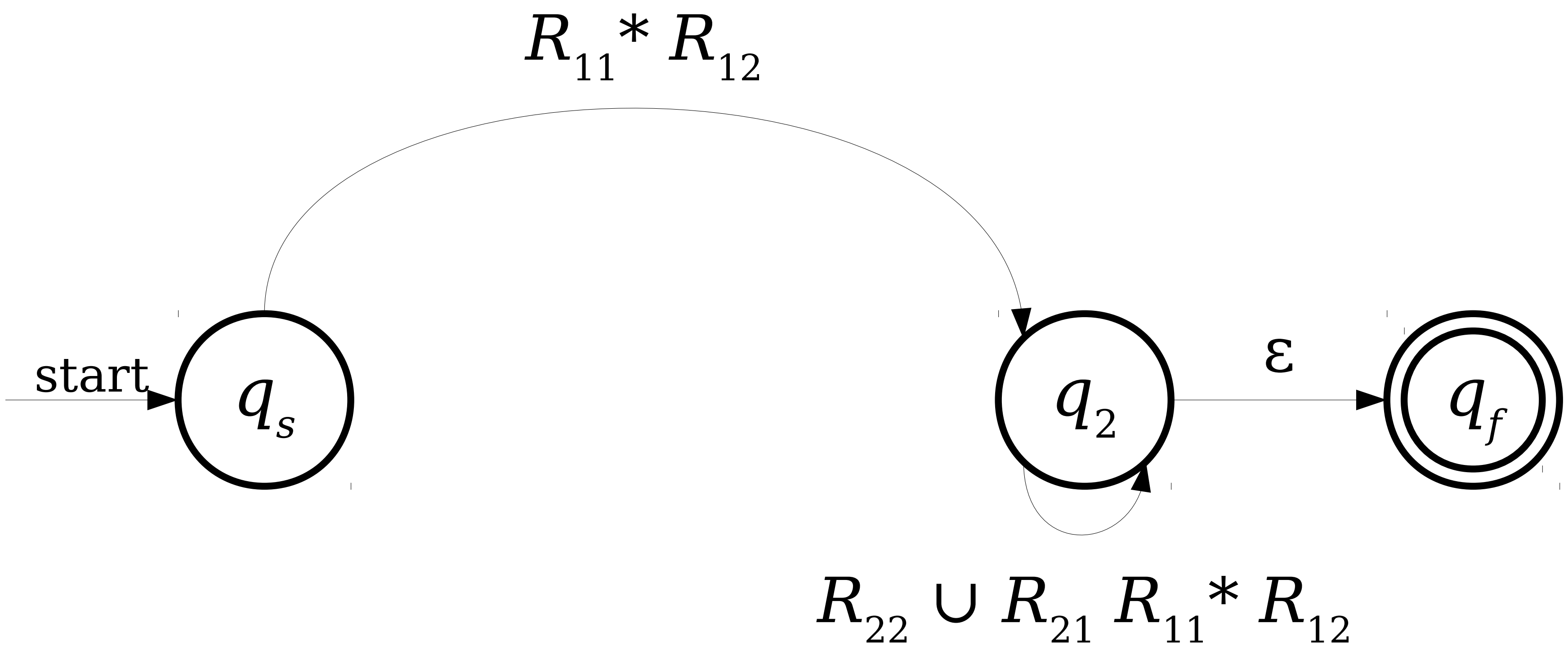
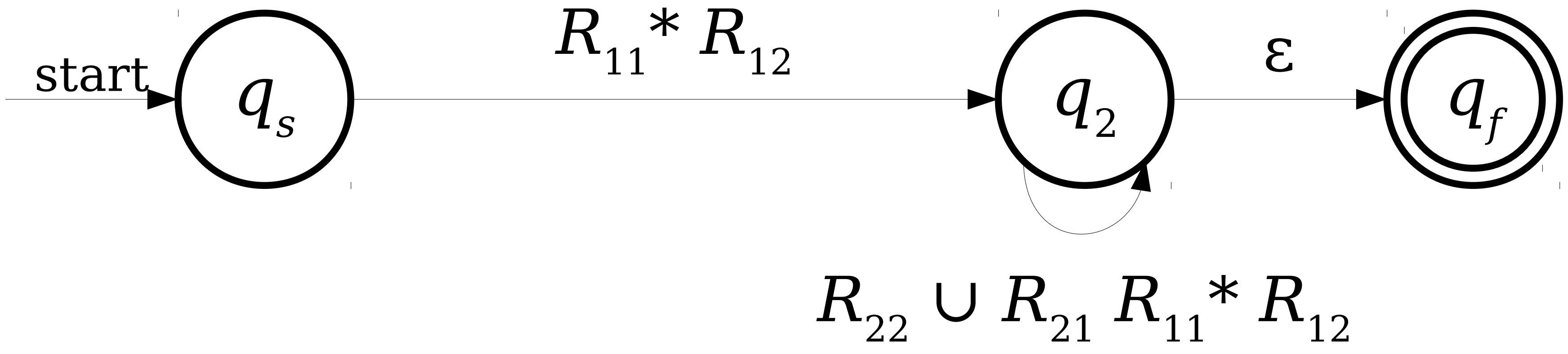



Note: We're using **union** to combine these transitions together.

# From NFAs to Regular Expressions



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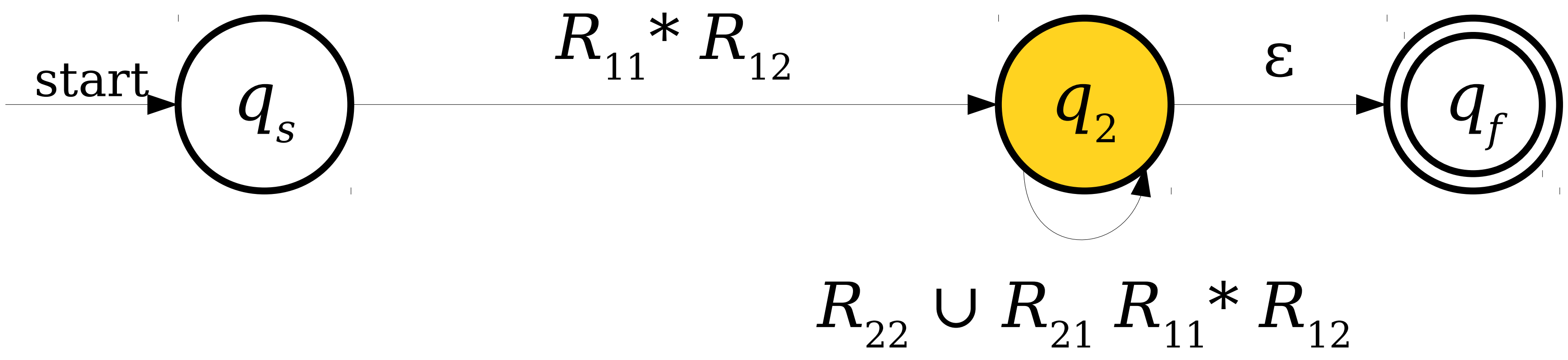

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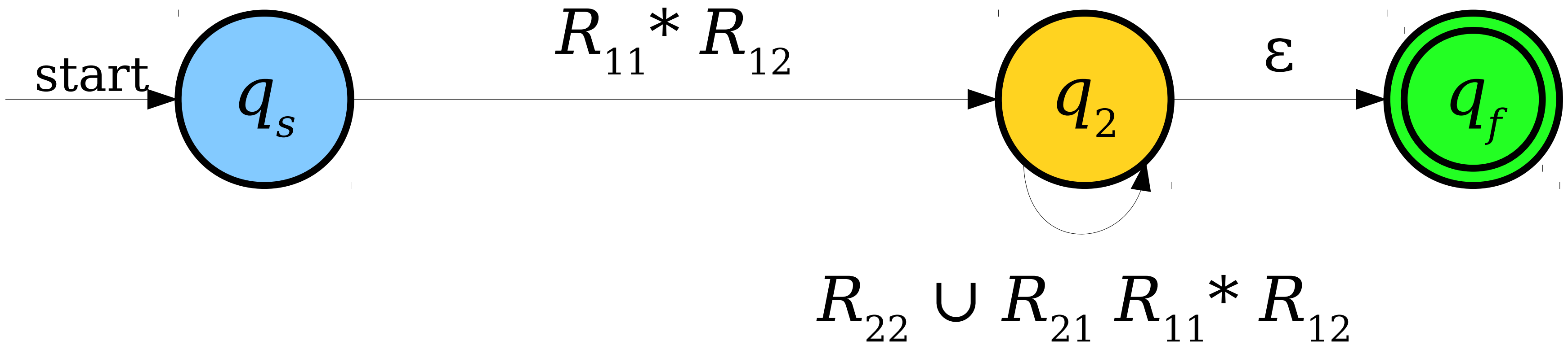

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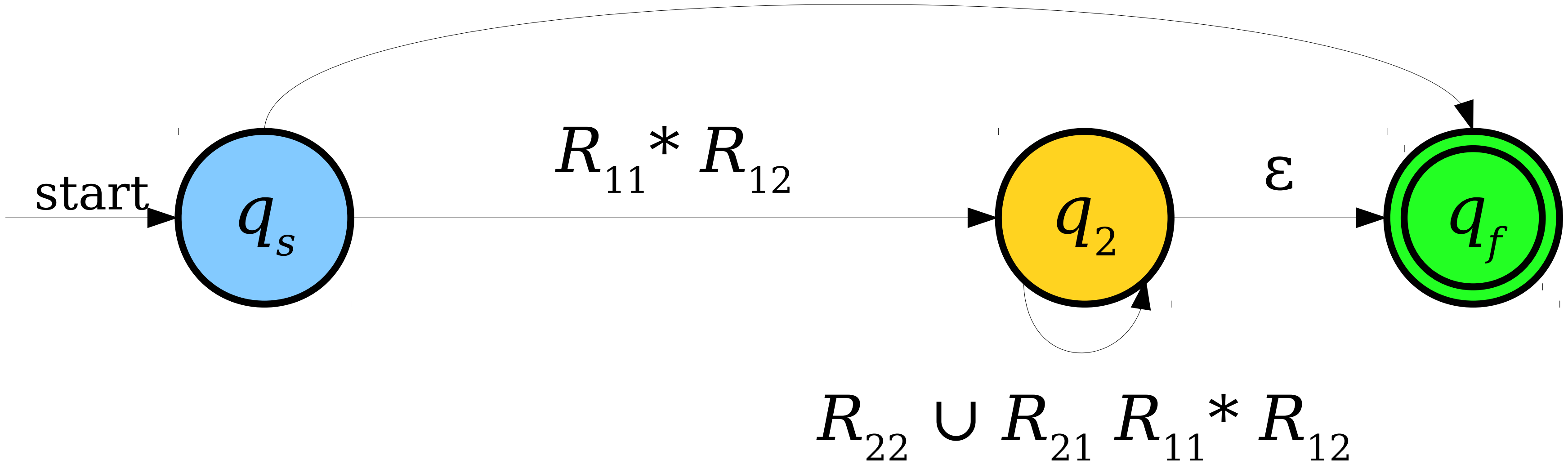
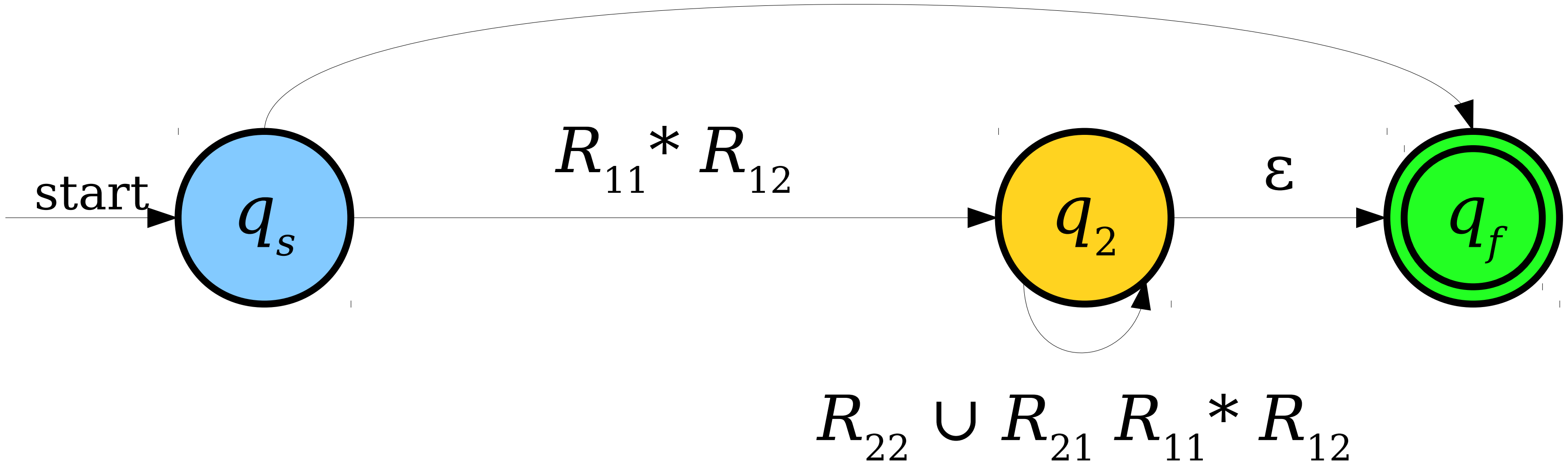


# From NFAs to Regular Expressions

**Quick check:** what goes on this transition?



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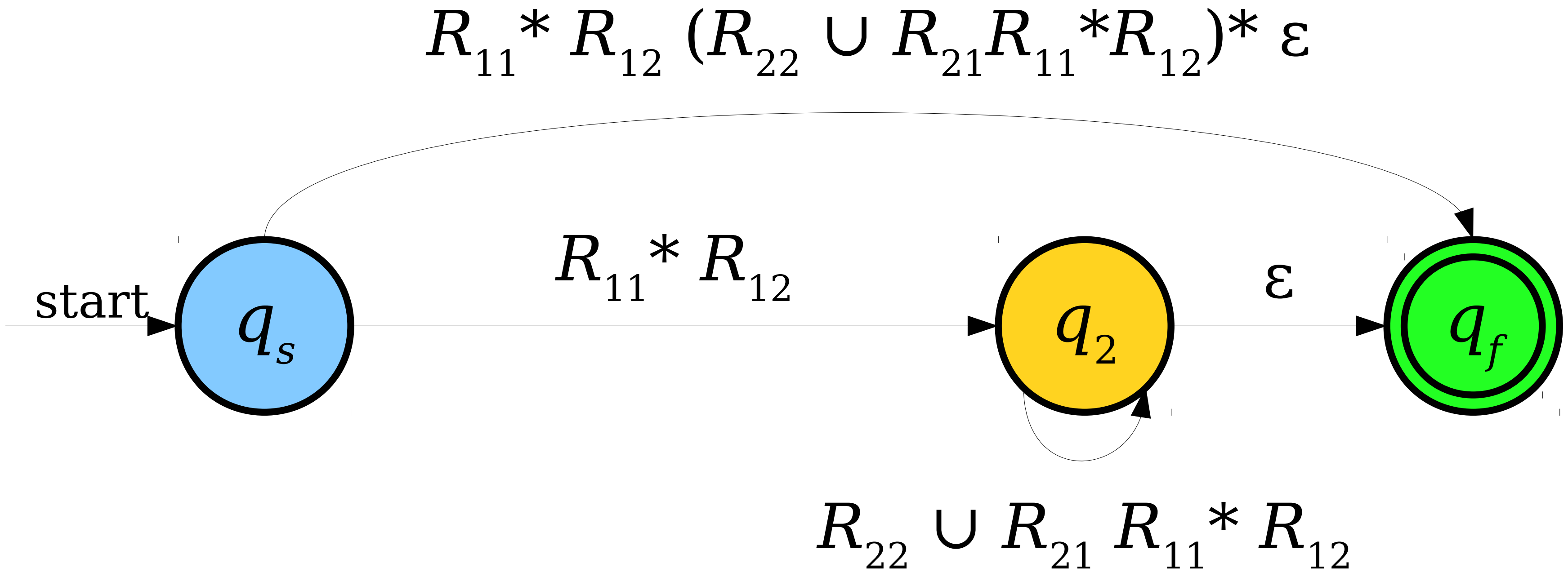

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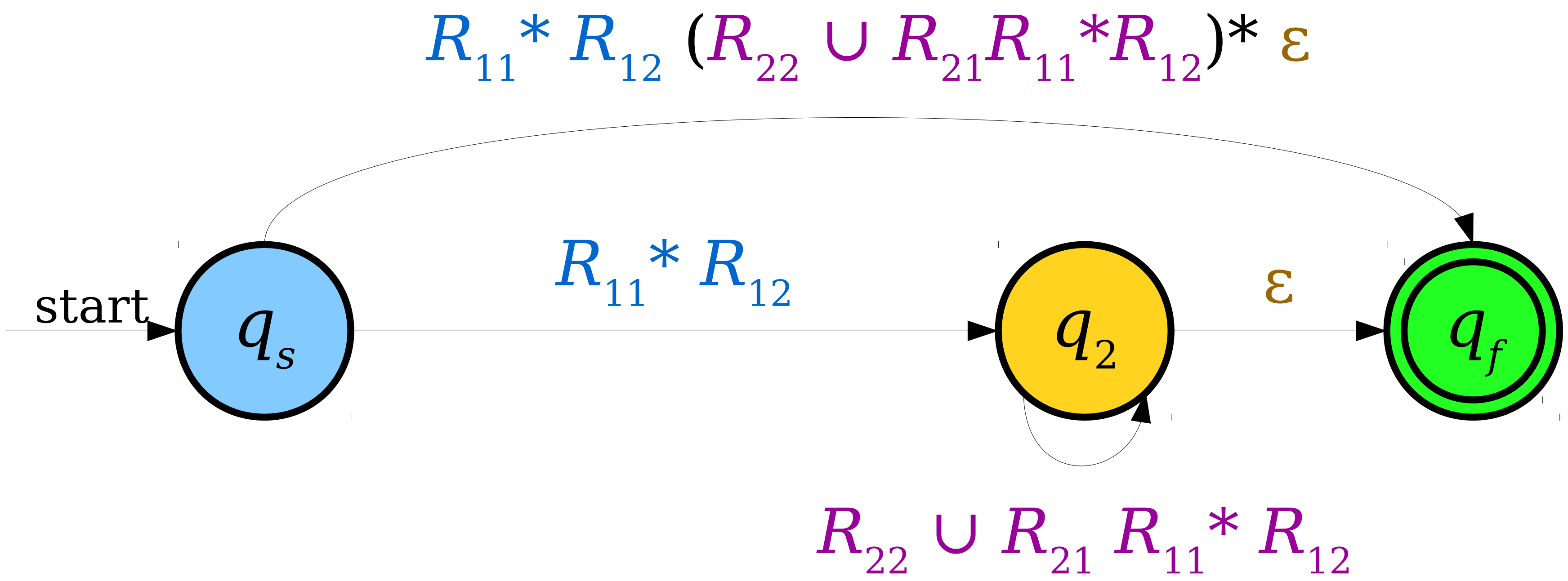

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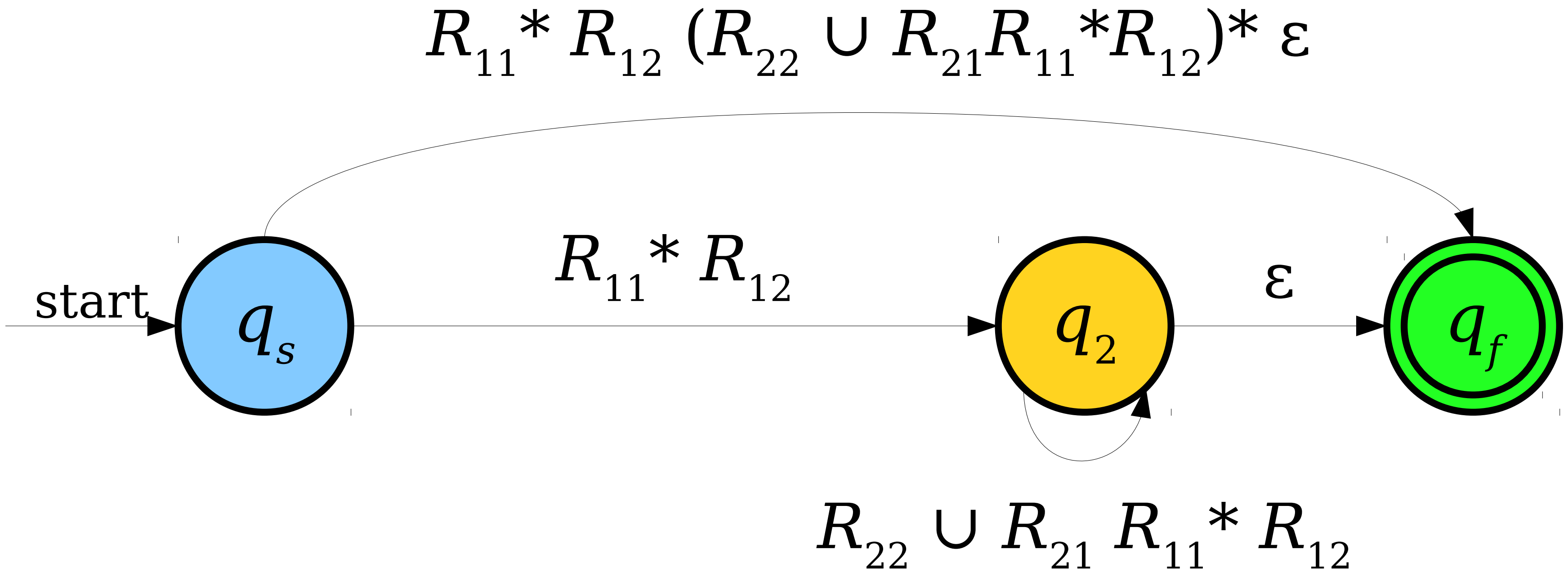

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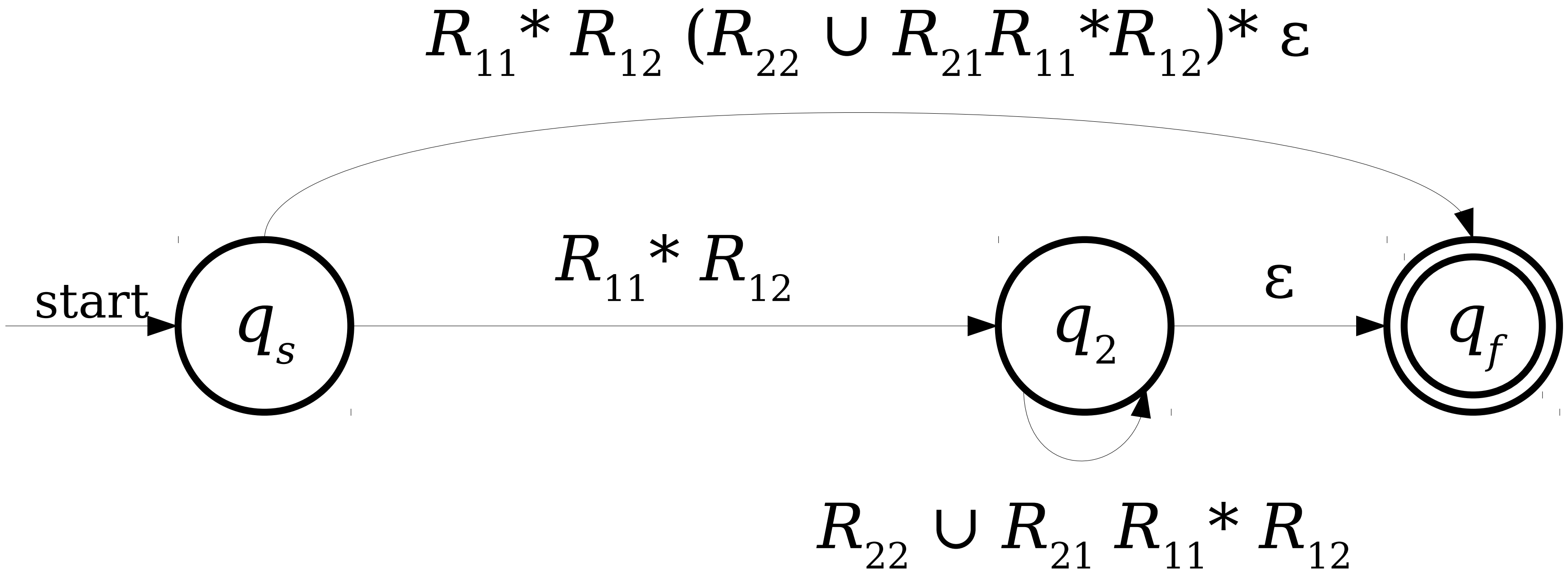

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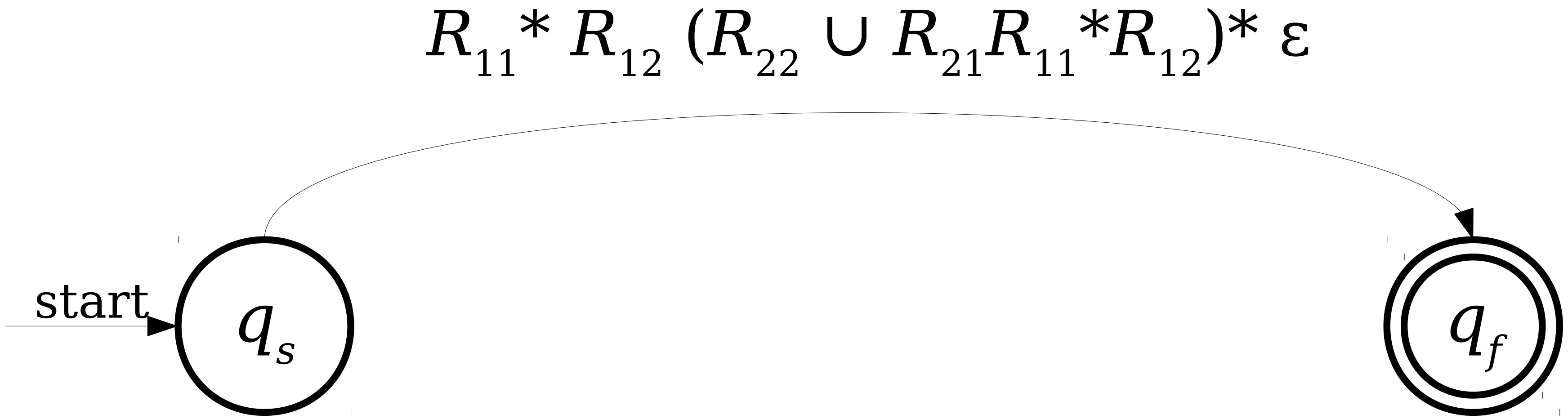

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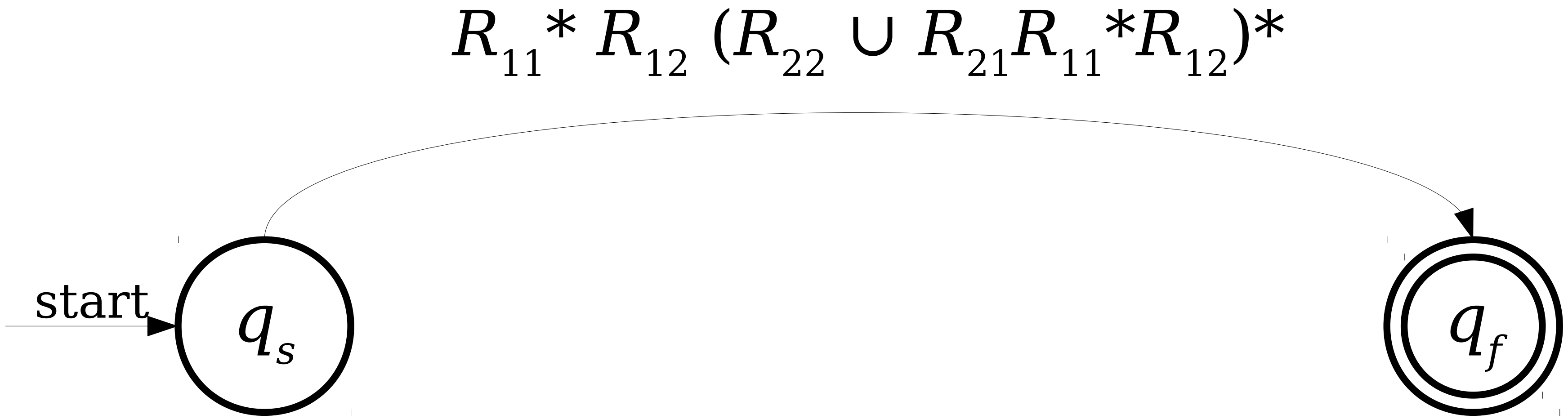

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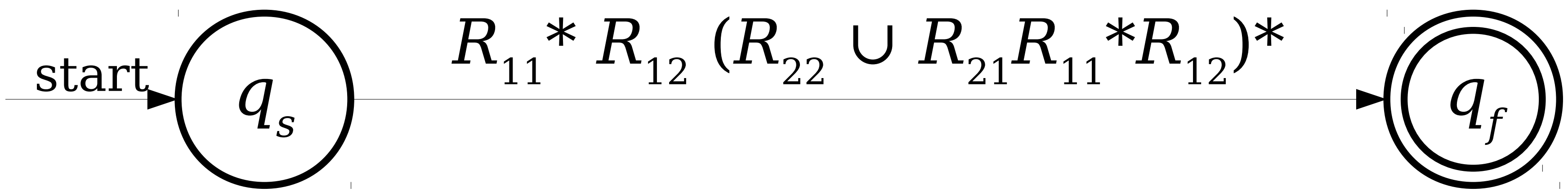

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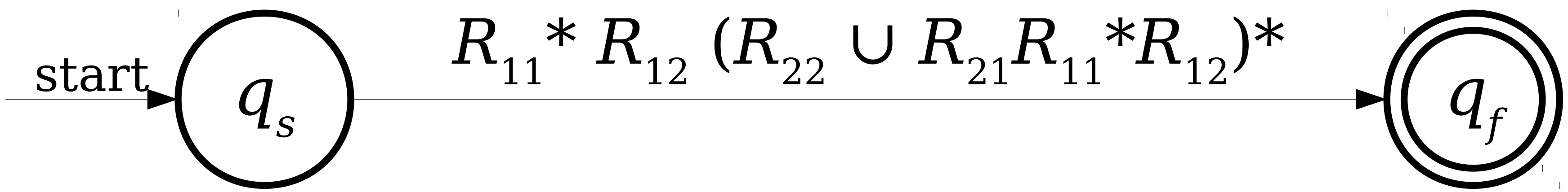
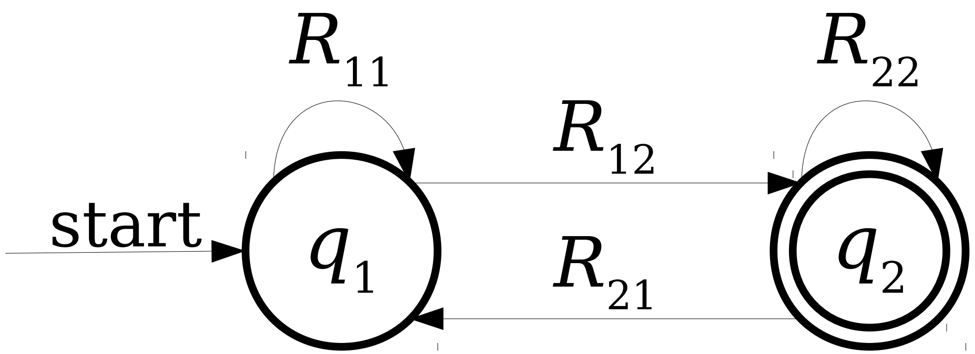


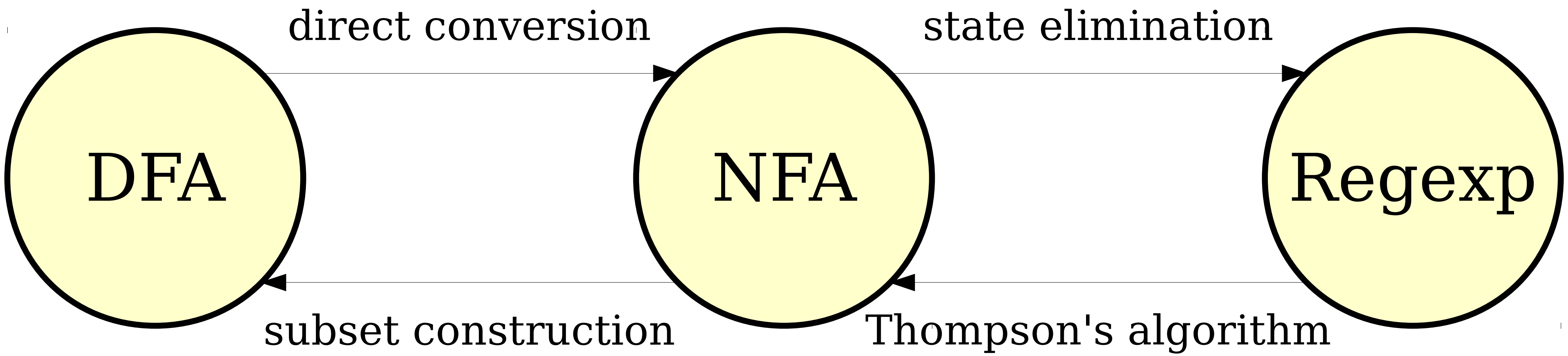
# The State-Elimination Algorithm

- Start with an NFA $N$ for the language $L$.
- Add a new start state $q_s$ and accept state $q_f$ to the NFA.
  - Add an ε-transition from $q_s$ to the old start state of $N$.
  - Add ε-transitions from each accepting state of $N$ to $q_f$, then mark them as not accepting.
- Repeatedly remove states other than $q_s$ and $q_f$ from the NFA by “shortcutting” them until only two states remain: $q_s$ and $q_f$.
- The transition from $q_s$ to $q_f$ is then a regular expression for the NFA.

# The State-Elimination Algorithm

- To eliminate a state $q$ from the automaton, do the following for each pair of states $q_0$ and $q_1$, where there's a transition from $q_0$ into $q$ and a transition from $q$ into $q_1$:
  - Let $R_{in}$ be the regex on the transition from $q_0$ to $q$.
  - Let $R_{out}$ be the regex on the transition from $q$ to $q_1$.
  - If there is a regular expression $R_{stay}$ on a transition from $q$ to itself, add a new transition from $q_0$ to $q_1$ labeled $((R_{in})(R_{stay})^*(R_{out}))$.
  - If there isn't, add a new transition from $q_0$ to $q_1$ labeled $((R_{in})(R_{out}))$
- If a pair of states has multiple transitions between them labeled $R_1, R_2, \ldots, R_k$, replace them with a single transition labeled $R_1 \cup R_2 \cup \ldots \cup R_k$.

# Our Transformations



***Theorem:*** The following are all equivalent:

- $L$ is a regular language.
- There is a DFA $D$ such that $\mathscr{L}(D) = L$.
- There is an NFA $N$ such that $\mathscr{L}(N) = L$.
- There is a regular expression $R$ such that $\mathscr{L}(R) = L$.

# Why This Matters

- The equivalence of regular expressions and finite automata has practical relevance.
  - Regular expression matchers have all the power available to them of DFAs and NFAs.
- This also is hugely theoretically significant: the regular languages can be assembled “from scratch” using a small number of operations!

# Next Time

- ***Applications of Regular Languages***
  - Answering “so what?”
- ***Intuiting Regular Languages***
  - What makes a language regular?
- ***The Myhill-Nerode Theorem***
  - The limits of regular languages.